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Period-Doubling and Neimark-Sacker Bifurcations in a Discrete Fractional Predator-Prey Model with Harvesting

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Abstract:

This paper investigates a discrete-time fractional-order predator-prey model with proportional harvesting on the prey population. The model is derived using the piecewise constant argument method, allowing the incorporation of memory effects in a discrete framework. The existence and local stability of equilibria are analyzed via the Jacobian matrix and characteristic equation. Conditions for period-doubling and Neimark-Sacker bifurcations are established, revealing transitions to periodic and quasi-periodic dynamics. Numerical simulations illustrate the impact of harvesting intensity on system stability and oscillatory behavior. The results highlight the significant role of harvesting and memory in shaping predator-prey dynamics.
Keywords: Population model; fractional derivative; harvesting; stability; bifurcation

1. Introduction

Predator-prey interactions are considered among the most basic processes that regulate the dynamics of ecological systems. Mathematical models of these interactions have provided important insights into population dynamics and species coexistence. Since the pioneering work of Lotka and Volterra [1, 2], many predator-prey models have been proposed and extended to include various biological processes such as prey refuge, fear effects, different types of functional responses, harvesting strategies, and cannibalism to better describe ecological interactions and population behaviors. These extensions have greatly enhanced the ecological realism of the classical predator-prey models and have led to various interesting dynamical behaviors such as stability switches, bifurcations, and chaos [3, 4, 5, 6, 7, 8, 9, 10].

In recent years, fractional calculus has gained significant attention in modeling biological and ecological systems. Fractional-order differential equations can include memory and hereditary aspects of biological processes that classical integer-order models cannot capture. In ecological systems, the current growth of a population may depend not only on its present state but also on its past states. Fractional derivatives provide a natural mathematical framework to describe these memory effects. Among the different definitions of fractional derivatives, the Caputo fractional derivative is commonly used in biological modeling because it allows for classical initial conditions and has a clear physical meaning. Fractional-order models have been effectively applied to many ecological and biological systems, revealing complex dynamic behaviors that are absent in integer-order models [11, 12, 13, 14, 15].

Mahmoudi and Eskandari [16] investigated the following fractional-order predator-prey model:

$$\begin{cases} \mathrm{D}^{\kappa}x(t) = x(t)(\alpha - \beta x(t)) - \gamma x(t)y(t), \\ \mathrm{D}^{\kappa}y(t) = \eta\gamma x(t)y(t) - \varepsilon y(t). \end{cases} \tag{1}$$

In system (1), $x(t)$ and $y(t)$ represent the densities of prey and predator populations at time t. The operator D^{κ} is the Caputo fractional derivative of order $0 < \kappa \leq 1$. The parameter α indicates the growth rate of the prey population when there are no predators. The parameter β is the intraspecific competition coefficient for the prey population. The parameter γ shows the predation rate coefficient, which describes the interaction between prey and predator populations. The parameter η indicates conversion efficiency, or how quickly the biomass of consumed prey turns into predator reproduction. The parameter ε represents the natural death rate of the predator population. All parameters are positive real numbers.

To understand the dynamics of system (1) using discrete-time techniques, Mahmoudi and Eskandari discretized the model with the piecewise constant argument method (PCAM) [17, 18, 19]. This method is commonly used for converting fractional differential systems into discrete systems while maintaining the fundamental behavior of the original system. Therefore, the next model is derived:

$$\begin{cases} x_{n+1} = x_n + \tau\Big(x_n(\alpha - \beta x_n) - \gamma x_n y_n\Big), \\ y_{n+1} = y_n + \tau\Big(\eta\gamma x_n y_n - \varepsilon y_n\Big), \end{cases} \tag{2}$$

where $\tau = \frac{h^{\kappa}}{\kappa\Gamma(\kappa)}$ and $h > 0$.

Harvesting is an important ecological factor that researchers have studied in population dynamics. In many biological systems, the population is subjected to harvesting processes such as fishing, hunting, pest control, and resource exploitation. Incorporating the harvesting process in predator-prey models is useful in understanding the impact of human activities on stability and survival. Research has been carried out on different types of harvesting processes, including constant harvesting, proportional harvesting, and nonlinear harvesting processes [20, 21, 22, 23, 24]. These research studies have shown the effect of harvesting on stability of equilibria.

Motivated by these observations, we extend model (1) by adding harvesting on the prey population. We obtain the following fractional-order system:

$$\begin{cases} \mathrm{D}^{\kappa}x(t) = x(t)(\alpha - \beta x(t)) - \gamma x(t)y(t) - \rho x(t), \\ \mathrm{D}^{\kappa}y(t) = \eta\gamma x(t)y(t) - \varepsilon y(t). \end{cases} \tag{3}$$

In system (3), the extra parameter $\rho > 0$ represents the harvesting rate of the prey population. The term $\rho x(t)$ describes proportional harvesting. This means that the number of harvested prey individuals is proportional to the current prey population density.

To explore the discrete dynamics of system (3), we utilise the piecewise constant argument technique. Let the initial conditions of system (3) be

$$x(0) = x_0, \qquad y(0) = y_0.$$

Then, by applying the piecewise constant approximation, system (3) can be written as

$$\begin{cases} {}^{C}D^{\kappa}x(t) = x\Big(\Big[\frac{t}{h}\Big]h\Big)\Big(\alpha - \beta x\Big(\Big[\frac{t}{h}\Big]h\Big)\Big) - \gamma x\Big(\Big[\frac{t}{h}\Big]h\Big)y\Big(\Big[\frac{t}{h}\Big]h\Big) - \rho x\Big(\Big[\frac{t}{h}\Big]h\Big), \\ {}^{C}D^{\kappa}y(t) = \eta\gamma x\Big(\Big[\frac{t}{h}\Big]h\Big)y\Big(\Big[\frac{t}{h}\Big]h\Big) - \varepsilon y\Big(\Big[\frac{t}{h}\Big]h\Big), \end{cases} \tag{4}$$

where $h > 0$ is the discretization step size and $[\cdot]$ denotes the greatest integer function.

Now consider $t \in [0, h)$. Then $\frac{t}{h} \in [0, 1)$, and hence $\Big[\frac{t}{h}\Big] = 0$. Therefore, system (4) reduces to

$$\begin{cases} {}^{C}D^{\kappa}x_1(t) = x_0(\alpha - \beta x_0) - \gamma x_0 y_0 - \rho x_0, \\ {}^{C}D^{\kappa}y_1(t) = \eta\gamma x_0 y_0 - \varepsilon y_0. \end{cases} \tag{5}$$

Using the standard solution formula for the Caputo fractional differential equation

$$^{C}D^{\kappa}u(t)=c,$$

where c is a constant, we obtain

$$u(t)=u(0)+\frac{t^{\kappa}}{\Gamma(\kappa+1)}c.$$

Thus, the solution of (5) is given by

$$\begin{cases} x_1(t)=x_0+\dfrac{t^{\kappa}}{\Gamma(\kappa+1)}\Big(x_0(\alpha-\beta x_0)-\gamma x_0y_0-\rho x_0\Big),\\ y_1(t)=y_0+\dfrac{t^{\kappa}}{\Gamma(\kappa+1)}\Big(\eta\gamma x_0y_0-\varepsilon y_0\Big). \end{cases} \tag{6}$$

Next, for $t\in[h,2h)$, we have $\dfrac{t}{h}\in[1,2)$, and so $\left[\dfrac{t}{h}\right]=1$. Hence, system (4) becomes

$$\begin{cases} ^{C}D^{\kappa}x_2(t)=x_1(h)\big(\alpha-\beta x_1(h)\big)-\gamma x_1(h)y_1(h)-\rho x_1(h),\\ ^{C}D^{\kappa}y_2(t)=\eta\gamma x_1(h)y_1(h)-\varepsilon y_1(h). \end{cases} \tag{7}$$

Again, applying the solution formula for the Caputo fractional derivative on the interval $[h,2h)$, we get

$$\begin{cases} x_2(t)=x_1(h)+\dfrac{(t-h)^{\kappa}}{\Gamma(\kappa+1)}\Big(x_1(h)\big(\alpha-\beta x_1(h)\big)-\gamma x_1(h)y_1(h)-\rho x_1(h)\Big),\\ y_2(t)=y_1(h)+\dfrac{(t-h)^{\kappa}}{\Gamma(\kappa+1)}\Big(\eta\gamma x_1(h)y_1(h)-\varepsilon y_1(h)\Big). \end{cases} \tag{8}$$

Proceeding in the same way, for $t\in[nh,(n+1)h)$, we have $\left[\dfrac{t}{h}\right]=n$. Therefore, system (4) takes the form

$$\begin{cases} ^{C}D^{\kappa}x_{n+1}(t)=x_n(nh)\big(\alpha-\beta x_n(nh)\big)-\gamma x_n(nh)y_n(nh)-\rho x_n(nh),\\ ^{C}D^{\kappa}y_{n+1}(t)=\eta\gamma x_n(nh)y_n(nh)-\varepsilon y_n(nh), \end{cases} \tag{9}$$

where $t\in[nh,(n+1)h)$. The solution of system (9) is then given by

$$\begin{cases} x_{n+1}(t)=x_n(nh)+\dfrac{(t-nh)^{\kappa}}{\Gamma(\kappa+1)}\Big(x_n(nh)\big(\alpha-\beta x_n(nh)\big)-\gamma x_n(nh)y_n(nh)-\rho x_n(nh)\Big),\\ y_{n+1}(t)=y_n(nh)+\dfrac{(t-nh)^{\kappa}}{\Gamma(\kappa+1)}\Big(\eta\gamma x_n(nh)y_n(nh)-\varepsilon y_n(nh)\Big), \end{cases} \tag{10}$$

for $t\in[nh,(n+1)h)$. Now, letting $t\to(n+1)h$ in (10), we obtain

$$\begin{cases} x_{n+1}\big((n+1)h\big)=x_n(nh)+\dfrac{h^{\kappa}}{\Gamma(\kappa+1)}\Big(x_n(nh)\big(\alpha-\beta x_n(nh)\big)-\gamma x_n(nh)y_n(nh)-\rho x_n(nh)\Big),\\ y_{n+1}\big((n+1)h\big)=y_n(nh)+\dfrac{h^{\kappa}}{\Gamma(\kappa+1)}\Big(\eta\gamma x_n(nh)y_n(nh)-\varepsilon y_n(nh)\Big). \end{cases} \tag{11}$$

For simplicity, by writing $x_n=x(nh),\ y_n=y(nh)$, system (11) can be rewritten as

$$\begin{cases} x_{n+1}=x_n+\dfrac{h^{\kappa}}{\Gamma(\kappa+1)}\Big(x_n(\alpha-\beta x_n)-\gamma x_ny_n-\rho x_n\Big),\\ y_{n+1}=y_n+\dfrac{h^{\kappa}}{\Gamma(\kappa+1)}\Big(\eta\gamma x_ny_n-\varepsilon y_n\Big). \end{cases} \tag{12}$$

Using the identity $\Gamma(\kappa+1)=\kappa\Gamma(\kappa)$, and defining $\tau=\frac{h^{\kappa}}{\kappa\Gamma(\kappa)}$, system (12) becomes

$$\begin{cases} x_{n+1}=x_n+\tau\Big(x_n(\alpha-\beta x_n)-\gamma x_ny_n-\rho x_n\Big),\\ y_{n+1}=y_n+\tau\Big(\eta\gamma x_ny_n-\varepsilon y_n\Big). \end{cases} \tag{13}$$

The inclusion of harvesting adds more ecological complexity to the system and may greatly affect long term dynamics of prey-predator interactions. Specifically, harvesting can change existence and stability properties of equilibria and may result in diverse behaviors like bifurcations and complex oscillations. Therefore, studying the effects of harvesting in fractional predator-prey systems is important for understanding how memory and resource exploitation together affect population dynamics. For further details on stability and bifurcation analysis of population models, we refer the reader to [25, 26, 27, 28].

The novelty of this paper consists in the extension of a fractional-order predator-prey model with the inclusion of proportional harvesting. These modifications result in a discrete-time dynamical system which retains memory properties and can be analyzed via bifurcations.

2. Stability analysis of equilibria

It is critical to understand the stability of fixed points when analyzing the predator-prey models since such points represent the system's equilibrium state. The determination of the stability or instability of equilibrium helps us determine the eventual future of the system. This is because, by using such knowledge, we will be able to determine whether the equilibrium state of species will persist or fail.

In order to investigate the local stability properties of the equilibrium, we make use of the tools like the Jacobian matrix and its corresponding characteristic equation. These are typical tools used in the analysis of the local stability properties of equilibriums in discrete dynamical systems.

The fixed points (x,y) for the system (13) can be obtained by solving

$$\begin{cases} x = x + \tau\Big(x(\alpha-\beta x) - \gamma xy - \rho x\Big), \\ y = y + \tau\Big(\eta\gamma xy - \varepsilon y\Big). \end{cases} \tag{14}$$

The discrete system (13) has three fixed points, which are

$$E_0 = (0,0),\ E_1 = (\frac{\alpha-\rho}{\beta},0),\ E^* = \left(\frac{\varepsilon}{\gamma\eta}, \frac{-\beta\varepsilon + \gamma\eta(\alpha-\rho)}{\gamma^2\eta}\right).$$

The trivial fixed point E_0 always exists. The boundary fixed point E_1 exists only when $\alpha > \rho$. The equilibrium E^* represents the unique positive fixed point provided that $\alpha > \rho$ and $\gamma > \frac{\beta\varepsilon}{\alpha\eta-\eta\rho}$. Since E^* is the only equilibrium with biological relevance, our analysis focuses on the classification of E^*, following the approach used in [29, 30].

Through simple calculations, we can find the Jacobian matrix at any fixed point (x,y) which is

$$J(x,y) = \begin{bmatrix} 1 + (\alpha - 2x\beta - y\gamma - \rho)\tau & -x\gamma\tau \\ y\gamma\eta\tau & 1 - \varepsilon\tau + x\gamma\eta\tau \end{bmatrix}.$$

The Jacobian matrix at E^* is

$$J(E^*) = \begin{bmatrix} 1 - \frac{\beta\varepsilon\tau}{\gamma\eta} & -\frac{\varepsilon\tau}{\eta} \\ -\frac{\beta\varepsilon\tau}{\gamma} + \eta(\alpha-\rho)\tau & 1 \end{bmatrix}. \tag{15}$$

The characteristic polynomial of $J(E^*)$ is used to find out the eigenvalues that will be responsible for the stability or any possible bifurcation near the equilibrium point. This can be expressed as

$$\Lambda(\xi) = \xi^2 + K_1\xi + K_0,$$

where

$$K_1 = -2 + \frac{\beta\varepsilon\tau}{\gamma\eta},$$
$$K_0 = 1 - \frac{\varepsilon\tau(\beta + \beta\varepsilon\tau + \gamma\eta(-\alpha+\rho)\tau)}{\gamma\eta}.$$

It follows from straightforward calculations that

$$\Lambda(0) = 1 - \frac{\varepsilon\tau(\beta + \beta\varepsilon\tau + \gamma\eta(-\alpha + \rho)\tau)}{\gamma\eta},$$
$$\Lambda(-1) = 4 + \varepsilon(\alpha - \rho)\tau^2 - \frac{\beta\varepsilon\tau(2 + \varepsilon\tau)}{\gamma\eta},$$
$$\Lambda(1) = -\frac{\varepsilon(\beta\varepsilon + \gamma\eta(-\alpha + \rho))\tau^2}{\gamma\eta}.$$

Because $\Lambda(1) > 0$, we get next result.

Theorem 2.1. *E^* is*

1. *a sink if* $4 + \varepsilon(\alpha - \rho)\tau^2 > \frac{\beta\varepsilon\tau(2+\varepsilon\tau)}{\gamma\eta}$ *and* $\varepsilon\tau(\beta + \beta\varepsilon\tau + \gamma\eta(-\alpha + \rho)\tau) > 0$*,*
2. *a source if* $4 + \varepsilon(\alpha - \rho)\tau^2 > \frac{\beta\varepsilon\tau(2+\varepsilon\tau)}{\gamma\eta}$ *and* $\varepsilon\tau(\beta + \beta\varepsilon\tau + \gamma\eta(-\alpha + \rho)\tau) < 0$*,*
3. *a saddle if* $4 + \varepsilon(\alpha - \rho)\tau^2 < \frac{\beta\varepsilon\tau(2+\varepsilon\tau)}{\gamma\eta}$*,*
4. *non-hyperbolic and* (13) *experiences period-doubling (PD) bifurcation at E^* if*

$$\frac{\beta\varepsilon\tau}{\gamma\eta} \neq 2,4 \quad \textit{and} \quad 4 + \varepsilon(\alpha - \rho)\tau^2 = \frac{\beta\varepsilon\tau(2 + \varepsilon\tau)}{\gamma\eta},$$

5. *non-hyperbolic and* (13) *experiences Neimark-Sacker (NS) bifurcation at E^* if*

$$0 < \frac{\beta\varepsilon\tau}{\gamma\eta} < 4 \quad \textit{and} \quad \beta + \beta\varepsilon\tau + \gamma\eta(-\alpha + \rho)\tau = 0.$$

3. Numerical examples

To further illustrate the results, we perform numerical simulations of system (13). We also consider how the parameter ρ, measuring the degree to which the prey is harvested, affects the stability of the coexistence equilibrium. This parameter also leads to different dynamics, including periodic and quasi-periodic oscillations.

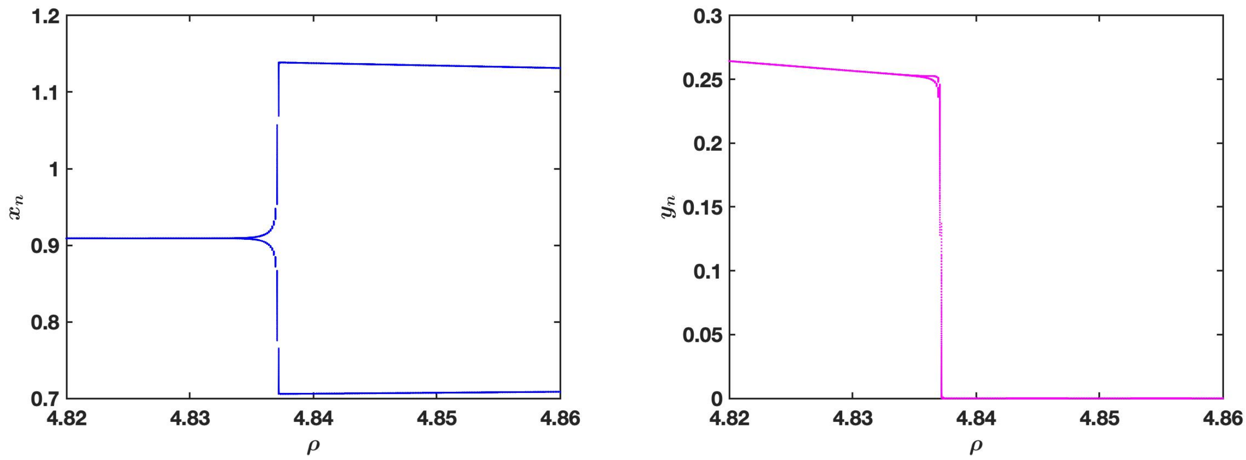
The numerical simulations have been done by iterating the system (13) according to a usual forward iteration procedure. With the chosen parameters and initial condition, the system has been iterated to a sufficiently large number of times such that any transient effects are removed and further iterations can be utilized to plot bifurcations diagrams and phase diagrams. All simulations have been done using MATLAB.

3.1. PD bifurcation

We fix the values: $\alpha = 9.8$, $\beta = 5.1$, $\gamma = 1.3$, $\varepsilon = 2.6$, $\eta = 2.2$, and $\tau = 0.45$. With these parameters, we have that the period doubling bifurcation occurs at $\rho = 4.83557$. Moreover, we have $E^* = (0.909091, 0.252359)$, and that the eigen-values of $J(E^*)$ are $\xi_1 = -1$ and $\xi_2 = 0.913636$. These results verify that the period doubling bifurcation occurs, as shown in Theorem 2.1. Figures 1a and 1b show the bifurcation diagrams when ρ takes values between 4.82 and 4.86. These figures verify that the system (13) experiences a PD bifurcation at $\rho = 4.83557$.

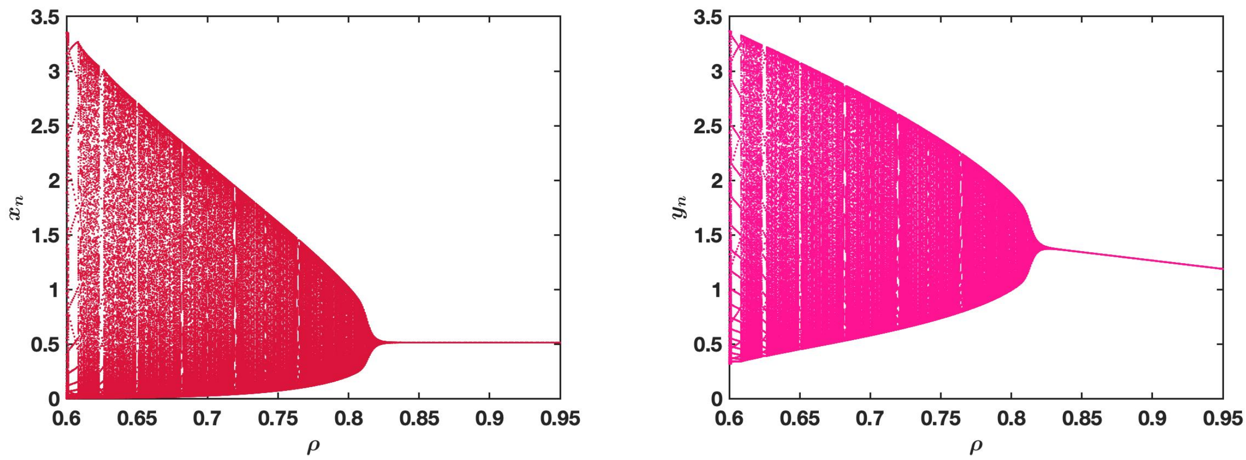
From a practical point of view, the appearance of period-doubling bifurcation implies that there is no stabilization of the equilibrium of coexistence for both predator and prey species populations. In other words, the populations start fluctuating between two different levels for each following period of time. This phenomenon can reflect either the seasonal or generational variations in ecosystems.

In addition, from the obtained simulation data, it is possible to conclude that even a slight variation of the value of the harvesting parameter ρ can cause the loss of stability and periodic behavior. This means that an incorrect harvesting policy may cause such dynamics in the populations' sizes.

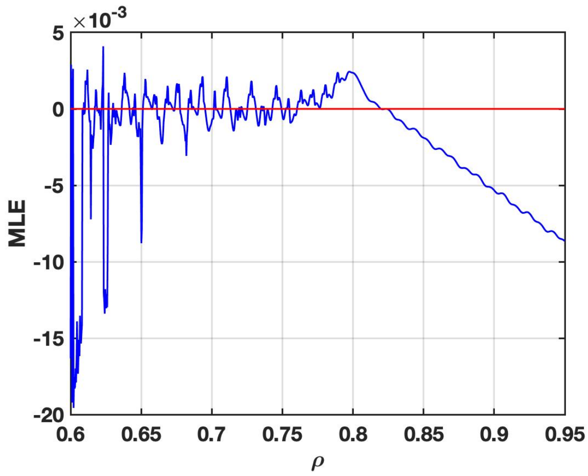


(a) **(b)**

Figure 1: Bifurcation plots of (13) with respect to ρ. Fixed parametric values are $\alpha = 9.8, \beta = 5.1, \gamma = 1.3, \varepsilon = 2.6, \eta = 2.2, \tau = 0.45$. The initial values are $x_0 = 0.9, y_0 = 0.3$.



(a) **(b)**



(c)

Figure 2: Bifurcation plots and MLE graph of (13) with respect to the parameter ρ. The remaining parameters are fixed as $\alpha = 1.85, \beta = 0.25, \gamma = 0.65, \varepsilon = 0.15, \eta = 0.45, \tau = 0.95$, and the initial conditions are taken as $x_0 = 0.5, y_0 = 1.4$.

3.2. NS bifurcation

We set $\alpha = 1.85, \beta = 0.25, \gamma = 0.65, \varepsilon = 0.15, \eta = 0.45$, and $\tau = 0.95$. For this setting, we have that bifurcation value of the NS bifurcation is given by $\rho = 0.82211$. Moreover, we have that the positive fixed point is $E^* = (0.512821, 1.38413)$, while the eigenvalues of $J(E^*)$ are $\xi_{1,2} = 0.939103 \pm 0.343637i$, with unit modulus. Thus, the NS bifurcation occurs at $\rho = 0.82211$, confirming Theorem 2.1. Bifurcation diagrams appear in Figures 2a and 2b for $\rho \in [0.60, 0.95]$. The related maximum Lyapunov exponent (MLE) plot is shown in Figure 2c. The negative values of MLE indicate stability or quasi-periodic behaviour.

To further clarify this dynamic change, Figures 3a to 3i display the phase portraits of (13) under various values of ρ. From these figures, we can see that if ρ is larger than its critical value 0.82211, then the coexistence equilibrium E^* is a stable sink to which all trajectories converge. As ρ approaches its critical value, the stability of E^* decreases gradually. Once ρ exceeds its critical value, a type of closed orbit appears around E^*, indicating that an NS bifurcation happens.

The Neimark-Sacker bifurcation corresponds to the change in dynamics from the stable coexistence to quasi-periodicity. The latter means that the population undergoes oscillatory changes in its size. However, it happens not regularly but rather in a complex yet bounded way. In particular, a decrease in the harvesting parameter value leads to higher amplitude oscillations and more irregular population dynamics. In other words, the reduction in the harvesting rate allows the growth of the population sizes but also leads to their irregular change. Thus, controlling the harvest is necessary in order to keep the population stable.

4. Conclusion

In this work, we considered a model for a predator and prey system with proportional harvesting on the prey population. We used the piecewise constant argument method to transform the continuous fractional system into a discrete system. This allowed us to analyze the behavior of the system using discrete systems. We determined the equilibrium points of the system and analyzed the stability of the coexistence equilibrium using the Jacobian matrix and its characteristic polynomial.

Analytical conditions were derived for different types of stability. It was demonstrated that changes in the harvesting parameter can cause major changes in the system's behavior. For example, stability loss of the coexistence equilibrium is possible through period doubling and NS bifurcations. Numerical simulations were used to validate the analytical finidings and to demonstrate how strength of harvesting influences the emergence of oscillations.

The results suggest that harvesting plays a crucial role in controlling population dynamics. Appropriate harvesting levels can stabilize the ecosystem, while excessive or insufficient harvesting may lead to oscillatory or unstable behaviors. These findings provide useful insights for ecological management, particularly in fisheries and wildlife conservation, where maintaining stable population levels is essential.

In comparison with traditional predator and prey models without harvesting and memory, the presented model provides a more comprehensive framework for understanding the impact of resource use and memory on the system. As a further direction, the presented model may be extended in various ways, for example, by considering predator harvesting, nonlinear harvesting, and other ecological aspects, such as refuge, fear, and cannibalism.

Data Availability: All data available in the manuscript.
Conflicts of Interest: On behalf of all authors, the corresponding author states that there is no conflict of interest.

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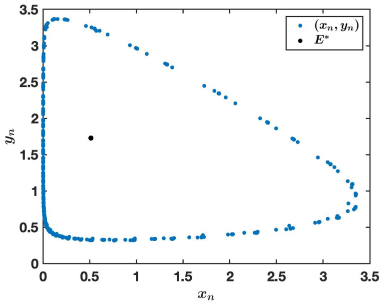
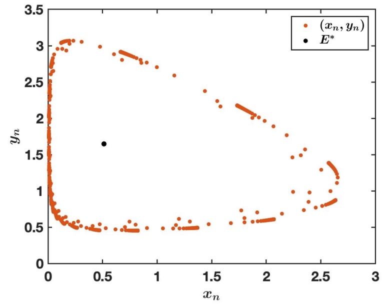
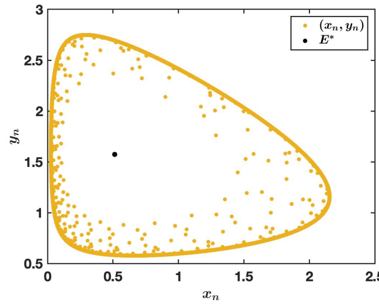


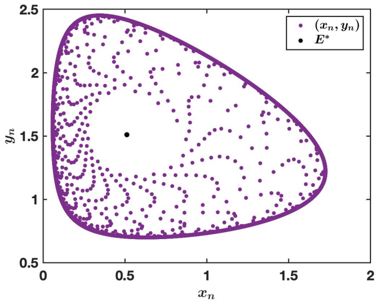
(a) $\rho = 0.60$



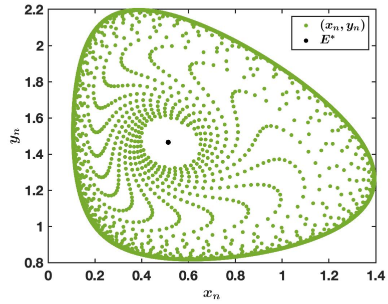
(b) $\rho = 0.65$



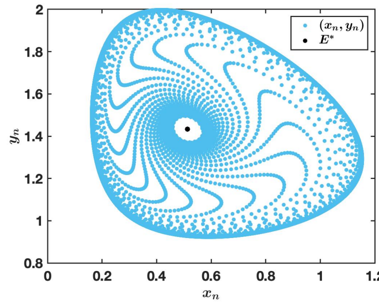
(c) $\rho = 0.70$



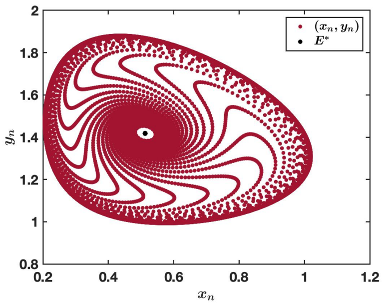
(d) $\rho = 0.74$



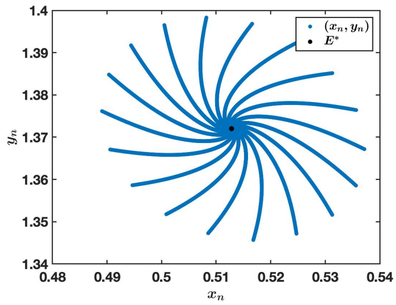
(e) $\rho = 0.77$



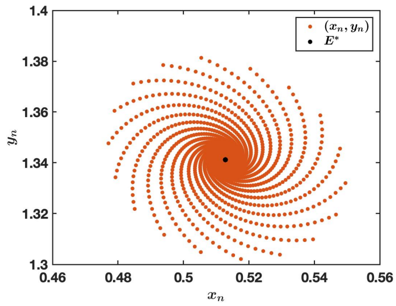
(f) $\rho = 0.79$



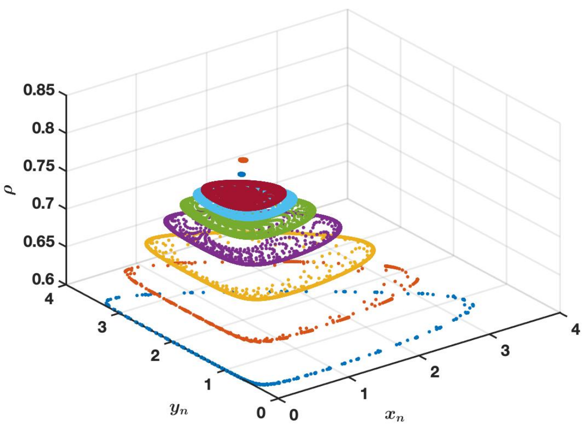
(g) $\rho = 0.80$



(h) $\rho = 0.83$



(i) $\rho = 0.85$



(j)

Figure 3: (a-i) Phase portraits of (13) corresponding to different values of ρ, with parameters fixed as $\alpha = 1.85, \beta = 0.25, \gamma = 0.65, \varepsilon = 0.15, \eta = 0.45, \tau = 0.95$ and initial conditions $x_0 = 0.5, y_0 = 1.4$. (j) Three-dimensional representation of the phase portraits.

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