Journal of Mathematical Modeling and Fractional Calculus

Website: https://acspublisher.com/journals/index.php/jmmfc

Received:26-11-2025, **Revised:** 03-02-2026
Accepted:01-03-2026, **Publication:** 20-06-2026
https://doi.org/ 10.48165/jmmfc.2026.3103, 3(1), 43-62

Functional Quantum Calculus: Theory and Applications via $q(t)$ and $h(t)$ Operators

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Abstract: This paper introduces a novel framework for functional quantum calculus based on generalized $q(t)$ and $h(t)$ calculi, with applications to geometric function theory and special functions. We define $q(t)$ – exponential, Gamma, Beta functions, hypergeometric series, Hermite polynomials, and starlike/convex functions, along with formulating $q(t)$- and $h(t)$-differential equations. Key contributions include establishing the relation between h(t)- and $q(t)$ – derivatives and the fundamental theorem of $h(t)$ – calculus. We also prove orthogonality for $q(\tau)$ – Hermite polynomials and provide numerical examples.

Keywords: : q – calculus; Differential equations; Geometric function theory; Special functions.

1. Intorduction

The q-calculus, often termed 'calculus without limits,' provides a discrete framework generalizing classical calculus through the parameter ‘q’. Recently, [12] expanded this theory by combining a time-dependent variable, introducing q(t)-calculus as a functional analogue of traditional q –calculus. This dynamic extension enables the modelling of temporally evolving systems, bridging discrete calculus with time-sensitive phenomena. Complementarily, [12] has investigated algebraic and number-theoretic properties of q -calculus.

As we know, the Quantum Calculus and special function [9, 14] plays a central role in mathematical and applied analysis as they are the solutions of differential equations, integral equations as well as variety of problems in physics. The Functional Quantum Calculus [12] provides even broader functional framework that unifies these generalized derivatives and integral operators under operator – theoretic perspectives. This approach not only enriches the theory of quantum special functions but also bridges operator theory, functional analysis and quantum group theory. Now, within Geometric Function Theory [15] the branch of complex analysis concerned with analytic and univalent functions. In recent developments [12], Functional Quantum Calculus will definitely provide a functional analytic foundation for developing the same.

This paper advances the theoretical framework of $q(t)$ - calculus and $h(t)$ – calculus by establishing novel properties and demonstrating their applications across multiple mathematical domains. In Section 2, we place a foundation for $q(t)$ – calculus and $h(t)$ - calculus, introducing essential definitions.

Section 3 presents theoretical advancements in three key areas which include differential equations, special functions, and geometric function theory, highlighting how these calculi enrich classical results. Section 4 further demonstrates the utility of these frameworks through concrete applications in the same domains. The paper concludes with a synthesis of the findings and references for further study.

Combining all these section one can say that, this paper develops a novel unified framework extending quantum calculus through generalized $q(t)$ and $h(t)$ calculi, bridging the gap between abstract theory and geometric function theory. The key novelty lies in introducing new function classes—including $q(t)$-starlike/convex functions, $q(t)$-hypergeometric series, and $q(\tau)$ −Hermite polynomials—while establishing for the first time the fundamental relationship between $h(t) - \mathit{and}\, q(t)$-derivatives. We prove the non – empty intersection Q and H, formulate $q(t)$ −differential equations, and establish the fundamental theorem of $h(t)$-calculus. Orthogonality properties and numerical examples validate our theoretical contributions.

2. Preliminaries:

In this section, we briefly present some basic definitions and properties of $\mathrm{q(t)} -$ calculus and $\mathrm{h(t)} -$ calculus. For more details see [2, 9, 12].

Definition 2.1: (**q – derivative**) Given an arbitrary function $f(x)$ then its $q(t)$ – derivative is denoted by $D_q f(x)$ and is given by [9]

$$D_q f(x) = \frac{f(qx)-f(x)}{qx-x} = \frac{f(qx)-f(x)}{(q-1)x} \quad (1)$$

Definition 2.2: (**Functional analogue of $q(t)$– derivative**) Given an arbitrary function $f(x)$ then it's $q(t)$ – derivative is denoted by $D_{q(t)}f(x)$ and is given by [12],

$$D_{q(t)}f(x) = \frac{f(q(t)x)-f(x)}{q(t)x-x} = \frac{f(q(t)x)-f(x)}{(q(t)-1)x}, \quad (2)$$

where, $Q = \{ q(t) \mid q(t) \to 1 \ as \ t \to 0 , t \in \mathbb{R} \}$

The above set is non – empty i. e. $Q \neq \emptyset$ and are well defined. e.g., If $q(t) = 1 + t , t \in \mathbb{R}$ then $q(t) \in Q$.

Definition 2.3: (**Functional analogue of $h(t)$– derivative**) Given an arbitrary function $f(x)$ then its $h(t)$ – derivative is denoted by $D_{h(t)}f(x)$ and is given by [12],

$$D_{h(t)}f(x) = \frac{f(x+h(t))-f(x)}{h(t)}, \quad (3)$$

where, $H = \{ h(t) \mid h(t) \to 0 \ as \ t \to 0 , t \in \mathbb{R} \}$

The above set is non – empty i.e. $H \neq \emptyset$ and are well defined. e.g., If $h(t) = t$, $t \in \mathbb{R}$ then $h(t) \in H$. Here onwards throughout the remaining part, $q(t) \in Q$ and $h(t) \in H$.

Definition 2.4: **(Functional analogue of $h(t)$ − Integral)** Given any arbitrary function $f(x)$ defined over [a, b] then its $h(t)$ − integral is defined using [9, 12],

$$\int_a^b f(x) d_{h(t)}x = \lim_{n\to\infty} h(t) \sum_{k=0}^{n-1} f(a + k \cdot h(t)), \quad where \ b = a + nh(t) \quad (4)$$

Definition 2.5: (**$q(t)$ – analogue of *n***) Given any positive integer *'n'* then its $q(t)$ – analogue is denoted by $[n]_{q(t)}$ and given by [12],

$$[n]_{q(t)} = \frac{[q(t)]^n-1}{q(t)-1} = 1 + q(t) + [q(t)]^2 + \cdots + [q(t)]^{n-1} \quad (5)$$

Definition 2.6: (**(t) – analogue of $n!$**) Given any positive integer *'n'* then its $q(t)$ – analogue is denoted by $[n!]_{q(t)}$ and given by [12],

$$[n!]_{q(t)} = [n]_{q(t)} \times [n-1]_{q(t)} \times \ldots \times [3]_{q(t)} \times [2]_{q(t)} \times [1]_{q(t)}$$

So, depending upon the choice of $q(t)$. The value of $[n!]_{q_1(t)}$ and $[n!]_{q_2(t)}$ varies. In other words, if $q_1(t) \to 1$ as $t \to 0$ much faster than $q_2(t) \to 1$ as $t \to 0$.

Then $[n!]_{q_1(t)}$ and $[n!]_{q_1(t)}$ tends much faster than to the value of $n!$ as $t \to 0$.

Definition 2.7: (**$q(t)$– shifting operator**) The $q(t)$ − shifting operator [12, 16] defined as

$$\Phi^a_{q(t)}(m) = q(t)m + (1 - q(t))a \quad (6)$$

where, $D_{\Phi^a_{q(t)}(m)}f(x) = \dfrac{f\left(\Phi^a_{q(t)}(m)x\right)-f(x)}{(\Phi^a_{q(t)}(m)-1)x}$. (7)

From the definition (4) and (5) one can easily see that,

1) $D_{\Phi^0_{q(t)}(1)}f(x) = D_{mq(t)}f(x)$.
2) $D_{\Phi^a_{q(t)}(a)}f(x) = D_a f(x)$ for $a > 0$.

3) $D_{\Phi^{m}_{q(t)}(m)} f(x) = D_m f(x)$ for $m > 0$.

4) $\Phi^{a}_{q(t)}(m) \rightarrow {}_{a}\Phi_q(m)$ as $t \rightarrow 0$.

The definition (6) satisfies the property that;

$$\Phi^{a}_{q_1(t)+q_2(t)}(m) = \{q_1(t) + q_2(t)\}m + (1 - \{q_1(t) + q_2(t)\})a$$

$$\Rightarrow \Phi^{a}_{q_1(t)+q_2(t)}(m) = \{q_1(t)\}m + \{q_2(t)\}m + \big(1 - q_1(t))a + (1 - q_2(t)\big)a$$

$$\Rightarrow \Phi^{a}_{q_1(t)+q_2(t)}(m) = \Phi^{a}_{q_1(t)}(m) + \Phi^{a}_{q_2(t)}(m)$$

Which means, the operator $\Phi^{a}_{q(t)}(m)$ is a linear operator

3. Theoretical Advancement:

Definition 3.1: ($q(t)$ – exponential function) The $q(t)$ – exponential function is given in terms of Hypergeometric function using [9],

$$\mathrm{e}_{q(t)}(\mathrm{z}) = \sum_{\mathrm{n}=0}^{\infty} \frac{\mathrm{z}^{\mathrm{n}}}{[\mathrm{n}]_{q(t)}!} = {}_0\phi_0(;;q(t),(1-q(t))\mathrm{z})$$

Here, $0 < q(t) < 1$ and $q(\tau) \in Q$.

Definition 3.2: ($q(\tau)$ – Gamma) The $q(\tau) -$ Gamma function is defined using [9],

$$\Gamma_{q(\tau)}(x) = \frac{(q(\tau);q(\tau))_{\infty}}{(1-q(\tau))^{x-1}} \sum_{n=0}^{\infty} \frac{[q(\tau)]^{n(x-1)}}{(q(\tau);q(\tau))_n},$$

Via $q(\tau)$ – integral it is given by,

$$\Gamma_{q(\tau)}(x) = \int_0^{\infty} t^{x-1}\, E_{q(\tau)}(-t)\, d_{q(\tau)}t$$

Here, $\mathrm{E}_{q(\tau)}(\mathrm{z})$ is the $q(\tau)$ – exponential function and $0 < q(\tau) < 1$ and $q(\tau) \in Q$.

Definition 3.3: ($q(\tau)$ –Beta Functions) The $q(\tau) -$ Beta function is defined using [9],

$$B_{q(\tau)}(x,y) = \frac{\Gamma_{q(\tau)}(x)\Gamma_{q(\tau)}(y)}{\Gamma_{q(\tau)}(x+y)}$$

Via $q(\tau)$ – integral it is given by:

$$B_{q(\tau)}(x,y) = \int_0^{1} t^{x-1}\, (1-t)_{q(\tau)}^{y-1}\, d_{q(\tau)}t$$

Here, $(1-\mathrm{t})_{q(\tau)}^{\mathrm{y}-1}$ is the $q(\tau)$ – binomial expansion and $0 < q(\tau) < 1$ and $q(\tau) \in Q$.

Definition 3.4: ($q(\tau)$ – Hypergeometric series) The $q(\tau)$ – Hypergeometric series is defined using [9],

$$ {}_r\phi_s\left(\begin{matrix} a_1, a_2, \dots, a_r \\ b_1, b_2, \dots, b_s \end{matrix}; q(\tau), z\right) = \sum_{n=0}^{\infty} \frac{(a_1; q(\tau))_n \cdots (a_r; q(\tau))_n}{(b_1; q(\tau))_n \cdots (b_s; q(\tau))_n} \frac{\left[(-1)^n [q(\tau)]^{\binom{n}{2}}\right]^{1+s-r}}{(q(\tau); q(\tau))_n} z^n $$

Here, $(a; q(\tau))_n = \prod_{k=0}^{n-1}\left(1 - a[q(\tau)]^k\right)$, $(a; q(\tau))_0 = 1$ and $[q(\tau)]^k$ is the k^{th} power of $q(\tau)$ and $0 < q(\tau) < 1$ with and $q(\tau) \in Q$.

Definition 3.5: ($q(\tau)$ – Hermite polynomials) The $q(\tau)$ – Hermite polynomials $H_n(x; q(\tau))$ are defined by the generating function using [9] as,

$$ \sum_{n=0}^{\infty} H_n\,(x; q(\tau)) \frac{\tau^n}{[n]_{q(\tau)}!} = \exp_{q(\tau)}(2x\tau - \tau^2) $$

Here, $\exp_{q(\tau)}(z)$ is the $q(\tau)$ – exponential function and $0 < q(\tau) < 1$ and $q(\tau) \in Q$.

Definition 3.6: (Class of Analytic functions) Let $\mathbb{D} = \{z \in \mathbb{C}: |z| < 1\}$ be the unit disk. Let $\mathcal{A}$ be the class of analytic functions in $\mathbb{D}$ normalized by,

$$ f(z) = z + a_2 z^2 + a_3 z^3 + \cdots. $$

Then for $f \in \mathcal{A}$ with $D_{q(t)}f(0) = f'(0) = 1$

$$ D_{q(t)}f(z) = 1 + \sum_{n=2}^{\infty} [\,n]_{q(t)} a_n z^{n-1}. $$

Definition 3.7: ($q(t)$ – Starlike Function) A function $f \in \mathcal{A}$ is said to be $q(t)$ – starlike if

$$ Re.\left(\frac{zD_{q(t)}f(z)}{f(z)}\right) > 0, \quad z \in \mathbb{D}. $$

We denote the class of $q(t)$ – starlike functions by $\mathcal{S}^*_{q(t)}$.

Definition 3.: ($q(t)$ – Convex Function) A function $f \in \mathcal{A}$ is said to be $q(t)$ – convex if

$$ \mathrm{Re}\left(\frac{D_{q(t)}(zD_{q(t)}f(z))}{D_{q(t)}f(z)}\right) > 0, \quad z \in \mathbb{D}. $$

We denote the class of $q(t)$ – convex functions by $\mathcal{K}_{q(t)}$.

Definition 3.9: ($q(t)$ – Differentiable function) A function $f(x)$ is said to be $q(t) -$ differentiable n – times if its n^{th} $q(t) -$ derivative exist but not $(n+1)^{th}$ differentiable.

Definition 3.10: ($q(t)$ – Differential Equation) Given a function $f(x)$ then it's n^{th}order linear $q(t)$ – Differential Equation is given by,

$$ F\left(x, f(x), D_{q(t)}f, D^2_{q(t)}f, \dots, D^n_{q(t)}f\right) = 0 $$

Where

- $f(x)$ is n – times $q(t)$ -differentiable functions and $D_{q(t)}f(x) = \frac{f(q(t)x) - f(x)}{(q(t)-1)x}$

Along with I.C., $f(0) = C_0$, $D_{q(t)}f(0) = C_1 \dots, D^{n-1}_{q(t)}f(0) = C_{n-1}$

Definition 3.11: ($h(t)$ – Differentiable function) A function $f(x)$ is said to be $h(t) -$ differentiable n – times if its n^{th} $h(t) -$ derivative exist but not $(n+1)^{th}$ differentiable.

Definition 3.12: ($h(t)$ – Differential Equation) Given a function $f(x)$ then it's n^{th}order linear $h(t)$ – Differential equation is given by,

$$F\left(x, f(x), D_{h(t)}f, D_{h(t)}^2 f, \dots, D_{h(t)}^n f\right) = 0$$

where,

- $f(x)$ is n – times $h(t)$-differentiable functions and $D_{h(t)}f(x) = \frac{f(x+h(t))-f(x)}{h(t)}$
- Along with I.C., $f(0) = C_0, D_{h(t)}f(0) = C_1 \dots, D_{h(t)}^{n-1}f(0) = C_{n-1}$

Theorem 3.1 : Given any function $f(x)$ with $D_{h(t)}f(x)$ and $D_{q(t)}f(x)$ are the $h(t) -$ derivative and $q(t) -$ derivative of $f(x)$ then the derivatives are related through the exponential change of variables [9],

$$q(t) = e^{h(t)} \Leftrightarrow h(t) = \ln(q(t))$$

and

$$x = e^u \; along \; with \; f(x) = g(u)$$

Proof:

Step I] Given a function $f(x)$ its $q(t) -$ derivative is given by

$$D_{q(t)}f(x) = \frac{f(q(t)x) - f(x)}{(q(t) - 1)x}$$

Substitute $x = e^u$, $f(x) = g(u)$, and $q(t) = e^{h(t)}$

Step II]

$$D_{q(t)}f(x) = \frac{g(u + \ln q(t)) - g(u)}{(e^{h(t)} - 1)e^u} = \frac{g(u + h(t)) - g(u)}{(e^{h(t)} - 1)e^u}$$

Step III] Now $h(t) -$ derivative of $g(u)$ is given by

$$D_{h(t)}g(u) = \frac{g(u + h(t)) - g(u)}{h(t)}.$$

Step IV] From the equations in Step [II] and Step [III]

$$\mathrm{D}_{q(t)}\mathrm{f(x)} = \frac{\mathrm{g}(\mathrm{u}+h(t)) - \mathrm{g(u)}}{(\mathrm{e}^{h(t)} - 1)\mathrm{e}^{\mathrm{u}}} \frac{h(t)}{h(t)}$$

$$\Longrightarrow \quad D_{q(t)}f(x) = \frac{h(t)}{e^{h(t)}-1} \cdot \frac{D_{h(t)}g(u)}{e^u}$$

Thus

$$D_{q(t)}\, f(x) = \frac{h(t)}{e^{h(t)}-1} \frac{D_{h(t)}\, g(u)}{x}$$

Theorem 3.2: There does not exist any function $f(x)$ such that, $D_{h(t)}f(x) = D_{q(t)}f(x)$ unless and until $q(t)$ and $h(t)$ are related by the relation $q(t) = 1 + \frac{h(t)}{x_0}$ for $x_0 \neq 0$ along with $f(x)$ is linear or exponential with specific parameter constraints [9].

Proof:

Case I] $f(x) = y = mx + c$ then

$D_{h(t)}f(x) = \frac{f(x+h(t))-f(x)}{h(t)} = \frac{m(x+h(t))+c-(mx+c)}{h(t)}$

$\Rightarrow D_{h(t)}f(x) = \frac{mx+mh(t)+c-(mx+c)}{h(t)} = \frac{mh(t)}{h(t)} = m$ --- (3.2.1)

$D_{q(t)}f(x) = \frac{f(q(t)x)-f(x)}{(q(t)-1)x} = \frac{mq(t)x+c-(mx+c)}{(q(t)-1)x}$

$\Rightarrow D_{q(t)}f(x) = \frac{mq(t)x+c-(mx+c)}{(q(t)-1)x} = \frac{m(q(t)-1)x}{(q(t)-1)x} = m$ --- (3.2.2)

From equation (3.2.1) and (3.2.2)

$$D_{h(t)}f(x) = D_{q(t)}f(x), \quad \forall q(t) \in Q, \forall h(t) \in H$$

Case II] $f(x) = e^{kx}$ where k is any constant, then by definition

$$D_{q(t)}f(x) = \frac{e^{kq(t)x} - e^{kx}}{(q(t)-1)x} \text{ and } D_{h(t)}f(x) = \frac{e^{k(x+h(t))} - e^{kx}}{h(t)} = e^{kx} \cdot \frac{e^{kh(t)}-1}{h(t)}$$

Now if we equate LHS of both sides, we get

$D_{q(t)}f(x) = D_{h(t)}f(x)$

$\Leftrightarrow \frac{e^{kq(t)x} - e^{kx}}{(q(t)-1)x} = e^{kx} \cdot \frac{e^{kh(t)} - 1}{h(t)} \Leftrightarrow \frac{e^{kx(q(t)-1)} - 1}{(q(t)-1)x} = \frac{e^{kh(t)} - 1}{h(t)}$

by cancelling e^{kx} from both sides as it is non – zero term

For this to hold for all x, we must have:

$$k(q(t)-1)x = kh(t) \Rightarrow q(t) - 1 = \frac{h(t)}{x_0}, \quad \text{where } x_0 \triangleq \frac{1}{k} \ (k \neq 0)$$

Thus for $f(x) = e^{kx}$, equality holds if and only if $q(t) = 1 + \frac{h(t)}{x_0}, \quad x_0 = \frac{1}{k}$.

Theorem 3.3 (Fundamental Theorem of $\boldsymbol{h(t)}$ – Calculus): If $F(x)$ is an antiderivative of $f(x)$ under the $h(t)$ - derivative $D_{h(t)}$, meaning $D_{h(t)}F(x) = f(x)$, and if the limit $\lim_{s\to 0}F(s)$ exists, then for $0 < a < b$ [9],

$$\int_a^b f(s)d_{h(t)}s = \int_a^b D_{h(t)} F(s)d_{h(t)}s = F(b) - F(a)$$

where $0 < h(t) < 1$.

Proof:

Assume, $D_{h(t)}F(x) = f(x)$. By the definition of the $h(t)$-derivative:

$$f(\mathrm{x}) = \frac{F(x+h(t))-F(x)}{h(t)} \quad \text{--- (3.3.1)}$$

The $h(t)$ – integral of f from a to b using the definition **(IV)**,

$$\int_a^b f(x)d_{h(t)}x = \lim_{n\to\infty} h(t) \sum_{k=0}^{n-1} f(a + k \cdot h(t))$$

We substitute $f(a+k\cdot h(t))$ in equation (A) and get,

$$\Rightarrow\ f\big(\mathrm{a}+\mathrm{k}\cdot h(t)\big) = \frac{\mathrm{F}\big(\mathrm{a}+\mathrm{k}.h(t)+h(t)\big)-\mathrm{F}\big(\mathrm{a}+\mathrm{k}.h(t)\big)}{h(t)}$$

Which gives, the relation between $h(t)$ – integral of $f(x)$ and definition (IV)

$$\int_a^b f(x) d_{h(t)}x = \lim_{n\to\infty} h(t) \sum_{k=0}^{n-1} \frac{F\big(a+(k+1)h(t)\big)-F\big(a+k\cdot h(t)\big)}{h(t)}$$

$$= \lim_{n\to\infty} \sum_{k=0}^{n-1} [F(a+(k+1)h(t))-F(a+k\cdot h(t))]$$

$As\ h(t)$ get cancelled.

$$\Rightarrow\ \int_a^b f(x) d_{h(t)}x = \lim_{n\to\infty} \sum_{k=0}^{n-1} [F(a+(k+1)h(t))-F(a+k\cdot h(t))]$$

But, $\sum_{\mathrm{k}=0}^{\mathrm{n}-1}[\mathrm{F}(\mathrm{a}+(\mathrm{k}+1)h(t))-\mathrm{F}(\mathrm{a}+\mathrm{k}\cdot h(t))]$

$$= F\big(a+nh(t)\big)-F(a) = F(b)-F(a)$$

$\Rightarrow\ \int_a^b f(x) d_{h(t)}x = F(b)-F(a).$

Theorem 3.4: The $q(t)-$ derivative of $q(t)-$ Hypergeometric function defined above is given by [9],

$$D_{q(t),z}\, {}_r\phi_s \left({a_1,\dots,a_r \atop b_1,\dots,b_s}; q(t), z\right)$$

$$= \frac{\prod_{i=1}^{r}(1-a_i)}{\prod_{j=1}^{s}(1-b_j)} \cdot \frac{(-1)^{1+s-r}}{1-q(t)} \cdot {}_r\phi_s \left({a_1 q(t),\dots,a_r q(t) \atop b_1 q(t),\dots,b_s q(t)}; q(t), [q(t)]^{1+s-r} z\right)$$

where, $\mathrm{q(t)} \to 1\ as\ t \to 0$ and $0 < q(t) < 1$.

Proof:

Let us consider, $F(z) = {}_{\mathrm{r}}\phi_{\mathrm{s}}\left({\mathrm{a}_1,\dots,\mathrm{a}_\mathrm{r} \atop \mathrm{b}_1,\dots,\mathrm{b}_\mathrm{s}}; q(t), \mathrm{z}\right)$

$$\Rightarrow F(z) = \sum_{n=0}^{\infty} A_n z^n, \quad \text{where} \quad A_n = \frac{\prod_{\mathrm{i}=1}^{\mathrm{r}}(\mathrm{a_i}; q(t))_\mathrm{n}}{\prod_{\mathrm{j}=1}^{\mathrm{s}}(\mathrm{b_j}; q(t))_\mathrm{n}} \frac{\left[(-1)^\mathrm{n}[q(t)]^{\binom{\mathrm{n}}{2}}\right]^{1+\mathrm{s}-\mathrm{r}}}{(q(t);q(t))_\mathrm{n}}$$

Then $q(t)$ – derivative term by term gives,

$$D_{q(t),z}F(z) = \sum_{n=1}^{\infty} A_n\, [n]_{q(t)} z^{n-1}, \quad where \quad [n]_{q(t)} = \frac{1-[q(t)]^n}{1-q(t)}$$

Now, put $n = m+1$

$$D_{q(t),z}F(z) = \sum_{m=0}^{\infty} A_{m+1}\, [m+1]_{q(t)} z^m$$

Where

$$A_{m+1} = A_m \cdot \frac{\prod_{i=1}^{r}(1 - a_i[q(t)]^m)}{\prod_{j=1}^{s}(1 - b_j[q(t)]^m)} \cdot \frac{1}{1 - [q(t)]^{m+1}} \cdot (-1)^{1+s-r}[q(t)]^{m(1+s-r)}$$

Multiply by $[m+1]_{q(t)} = \frac{1-[q(t)]^{m+1}}{1-q(t)}$ on both sides of the equation, we get,

$$\mathrm{A}_{\mathrm{m}+1}[\mathrm{m}+1]_{\mathrm{q(t)}} = \mathrm{A}_{\mathrm{m}} \cdot \frac{\prod_{\mathrm{i}=1}^{\mathrm{r}}(1 - \mathrm{a}_{\mathrm{i}}[q(t)]^{\mathrm{m}})}{\prod_{\mathrm{j}=1}^{\mathrm{s}}(1 - \mathrm{b}_{\mathrm{j}}[q(t)]^{\mathrm{m}})} \cdot \frac{(-1)^{1+\mathrm{s}-\mathrm{r}}[q(t)]^{\mathrm{m}(1+\mathrm{s}-\mathrm{r})}}{(1 - q(t))}$$

Using the identity:

$$1 - a_i[q(t)]^m = (1 - a_i)\frac{(a_i q(t); q(t))_m}{(a_i; q(t))_m}$$

We get

$$A_{m+1}[m+1]_{q(t)} = \frac{\prod_{i=1}^{r}(1 - a_i)}{\prod_{j=1}^{s}(1 - b_j)} \cdot \frac{(-1)^{1+s-r}}{1 - q(t)} \cdot [q(t)]^{m(1+s-r)} \cdot \frac{\prod_{i=1}^{r}(a_i q(t); q(t))_m}{\prod_{j=1}^{s}(b_j q(t); q(t))_m} \cdot \frac{\left[(-1)^m [q(t)]^{\binom{m}{2}}\right]^{1+s-r}}{(q(t); q(t))_m}$$

$$\Rightarrow A_{m+1}[m+1]_{q(t)} = \frac{\prod_{i=1}^{r}(1-a_i)}{\prod_{j=1}^{s}(1-b_j)} \cdot \frac{(-1)^{1+s-r}}{1-q(t)} \cdot B_m$$

Where B_m is the coefficient of ${}_r\phi_s \left(\begin{matrix} a_1 q(t), \ldots, a_r q(t) \\ b_1 q(t), \ldots, b_s q(t) \end{matrix} ; q(t), [q(t)]^{1+s-r} z \right)$.

$$\Rightarrow D_{q(t),z}F(z) = \frac{\prod_{i=1}^{r}(1 - a_i)}{\prod_{j=1}^{s}(1 - b_j)} \cdot \frac{(-1)^{1+s-r}}{1 - q(t)} \cdot {}_r\phi_s \left(\begin{matrix} a_1 q(t), \ldots, a_r q(t) \\ b_1 q(t), \ldots, b_s q(t) \end{matrix} ; q(t), [q(t)]^{1+s-r} z \right)$$

Theorem 3.5: The property for $q(\tau) -$ Gamma function is given by [11],

$$\Gamma_{q(\tau)}(\mathrm{x}+1) = [\mathrm{x}]_{q(\tau)}\Gamma_{q(\tau)}(\mathrm{x})$$

Where $0 < q(\tau) < 1$ for the convergence and existence of the integral.

Proof: We Know that,

$$\Gamma_{q(\tau)}(\mathrm{x}) = \int_0^{\infty} \mathrm{t}^{\mathrm{x}-1}\, \mathrm{E}_{q(\tau)}(-\mathrm{t})\, \mathrm{d}_{q(\tau)}\mathrm{t} \Rightarrow \Gamma_{q(\tau)}(\mathrm{x}+1) = \int_0^{\infty} \mathrm{t}^{\mathrm{x}}\, \mathrm{E}_{q(\tau)}(-\mathrm{t})\, \mathrm{d}_{q(\tau)}\mathrm{t}$$

We know that [9]
[1] $\mathrm{E}_{q(\tau)}(-\mathrm{t}) = 1\,, 0 < q(\tau) < 1$
[2] $\mathrm{E}_{q(\tau)}(-\infty) = \lim_{t\to\infty} \frac{1}{\mathrm{E}_{q(\tau)}(-\mathrm{t})} = 0,\ 0 < q(\tau) < 1$
[3] Using $\mathrm{q}(\tau)$ integration by parts in [9]

We get,

$$\Gamma_{q(\tau)}(\mathrm{x}+1) = -\int_0^{\infty} \mathrm{t}^{\mathrm{x}}\, \mathrm{d}_{q(\tau)}(\mathrm{E}_{q(\tau)}(-\mathrm{t}))$$
$$= [x]_{q(\tau)} \int_0^{\infty} \mathrm{t}^{\mathrm{x}-1}\, \mathrm{E}_{q(\tau)}(-q(\tau)\mathrm{t})\, \mathrm{d}_{q(\tau)}\mathrm{t}$$

$$\Longrightarrow\ \Gamma_{q(\tau)}(\mathrm{x}+1)=[\mathrm{x}]_{q(\tau)}\Gamma_{q(\tau)}(\mathrm{x})$$

Theorem 3.6: The $q(\tau)$ – Hermite polynomials $\mathrm{H_n}(\mathrm{x};\mathrm{q}(\tau))$ satisfies the Orthogonality relation given by [11],

$$\int_{-\infty}^{\infty} H_m\,(x;q(\tau))H_n(x;q(\tau))w_{q(\tau)}(x)\,d_{q(\tau)}x=[n]_{q(\tau)}!\sqrt{\pi}\delta_{mn}$$

Where, $w_{q(\tau)}(x)=\frac{1}{\sqrt{\pi}}e_{q(\tau)}(-x^2)$ is the *q(τ)* – Gaussian weight.

Proof:

Case I] $\boldsymbol{n}=\mathbf{0}$

$$\mathrm{I}_{00}=\int_{-\infty}^{\infty} H_0\,(x)^2w_{q(\tau)}\,(x)d_{q(\tau)}\,x=\int_{-\infty}^{\infty} w_{q(\tau)}\ (x)d_{q(\tau)}\,x$$

It is known from the theory of q-integrals that

$$\int_{-\infty}^{\infty} e_{q(\tau)}\ (-x^2)d_{q(\tau)}\,x=[0]_{q(\tau)}\,!\sqrt{\pi}$$

Case II] $\boldsymbol{m\not\equiv n\ (mod\ 2)}$

The weight function $w_{q(\tau)}\,(x)$ is an even function and polynomials satisfies $H_n(-x;q(\tau)\,)=(-1)^nH_n(x;q(\tau)\,)$, so they have definite parity.

$\Longrightarrow H_m(x)H_n(x)$ is even if m and n have the same parity, and odd otherwise.

Since the *q(τ)* – integral over a symmetric interval of an odd function is zero, it follows that:

$$\int_{-\infty}^{\infty} H_m\,(x)H_n(x)w_{q(\tau)}\,(x)d_{q(\tau)}\,x=0$$

Case III] $\boldsymbol{m\equiv n\ (mod\ 2)}$

We will prove the result using induction on n, the $q(\tau)$ – Hermite polynomials satisfies the recurrence relation given by,

$$xH_n=H_{n+1}+[n]_{q(\tau)}\,H_{n-1}$$

$\Longrightarrow\qquad \int xH_mH_nw_{q(\tau)}d_{q(\tau)}x=I_{m,n+1}+[n]_{q(\tau)}I_{m,n-1}$ for any m --(3.6.1)

Similarly from the recurrence relation is given by,

$$xH_m=H_{m+1}+[m]_{q(\tau)}\,H_{m-1}$$

$\Longrightarrow\qquad \int xH_mH_nw_{q(\tau)}d_{q(\tau)}x=I_{m+1,n}+[m]_{q(\tau)}I_{m-1,n}$ for any n --(3.6.2)

From equation (3.6.1) and (3.6.2)

$I_{m,n+1}+[n]_{q(\tau)}I_{m,n-1}=\ I_{m+1,n}+[m]_{q(\tau)}I_{m-1,n}$ --(3.6.3)

By the induction hypothesis, $\mathrm{I_{kl}}=0$ for $\mathrm{k}\neq\mathrm{l}$ with $\mathrm{k},\mathrm{l}<\mathrm{n}$ in equation (3.6.3) we get

$$\mathrm{I_{mn}}=0\text{ for }\mathrm{m}\neq\mathrm{n}.$$

Hence, from Case I] to Case III]

$$I_{mn}=0\text{ for all }m\neq n.$$

Case IV] m = n

We know that, $q(\tau)$ – Hermite polynomials satisfies the recurrence relation given by

$$xH_n=H_{n+1}+[n]_{q(\tau)}H_{n-1}\Longrightarrow\ H_{n+1}=\ xH_n-\ +[n]_{q(\tau)}H_{n-1}$$

Squaring on both sides of the above equation gives;

$$H_{n+1}^2 = \left(xH_n - [n]_{q(\tau)}H_{n-1}\right)^2 = x^2H_n^2 - 2[n]_{q(\tau)}xH_nH_{n-1} + [n]_{q(\tau)}^2H_{n-1}^2.$$

Multiply by $w_{q(\tau)}(x)$ both sides and integrate:

$$I_{n+1,n+1} = \int x^2H_n^2w_{q(\tau)}d_{q(\tau)}x - 2[n]_{q(\tau)}\int xH_nH_{n-1}w_{q(\tau)}d_{q(\tau)}x + [n]_{q(\tau)}^2I_{n-1,n-1}.$$

Now, apply the recurrence for H_{n-1}:

$$xH_{n-1} = H_n + [n-1]_{q(\tau)}H_{n-2},$$

We find

$$\int xH_nH_{n-1}w_{q(\tau)}d_{q(\tau)}x = I_{n,n} + [n-1]_{q(\tau)}I_{n,n-2}.$$

But $I_{n,n-2} = 0$ by orthogonality, so:

$$\int xH_nH_{n-1}w_{q(\tau)}d_{q(\tau)}x = I_{n,n}.$$

Similarly, using the recurrence for H_n, we can express $\int x^2H_n^2w_{q(\tau)}d_{q(\tau)}x$ in terms of $I_{n,n}$ and $I_{n-1,n-1}$.

Substituting back and simplifying gives:

$$I_{n+1,n+1} = [n+1]_{q(\tau)}I_{n,n}.$$

Since, $I_{00} = \sqrt{\pi}$, it follows by induction that:

$$I_{n,n} = [n]_{q(\tau)}!\,\sqrt{\pi}.$$

Theorem 3.7: If $f(z) \in \mathcal{S}_{\mathrm{q(t)}}^*$, then for $n \geq 2$:

$$|a_n| \leq \prod_{k=0}^{n-2}\frac{[k+1]_{q(t)}+1}{[n-1]_{q(t)}!}.$$

Where, $f(z) = \sum_{n=1}^{\infty} a_n\,z^n$.

Proof:

Since $f \in \mathcal{S}_{q(t)}^*$, we have:

$$\frac{zD_{q(t)}f(z)}{f(z)} = 1 + \sum_{n=1}^{\infty} b_n\,z^n \text{ with } Re.\left(1 + \sum_{n=1}^{\infty} b_n\,z^n\right) > 0.$$

By Carathéodory's lemma: $|b_n| \leq 2$

Now, from the series expansion,

$$zD_{q(t)}f(z) = z + \sum_{n=2}^{\infty}[n]_{q(t)}a_nz^n \; and f(z) = z + \sum_{n=2}^{\infty} a_n\,z^n.$$

$$\Rightarrow \left(z + \sum_{n=2}^{\infty}[n]_{q(t)}a_nz^n\right) = \left(z + \sum_{n=2}^{\infty} a_n\,z^n\right)\left(1 + \sum_{n=1}^{\infty} b_n\,z^n\right)$$

Equating coefficients:

- For z^2: $[2]_{q(t)}a_2 = a_2 + b_1 \Rightarrow b_1 = ([2]_{q(t)} - 1)a_2$

- For z^3: $[3]_{q(t)}a_3 = a_3 + a_2b_1 + b_2$
- In general: $[n]_{q(t)}a_n = a_n + \sum_{k=1}^{n-1} a_{n-k}\, b_k$

Solving recursively and using $|b_k| \le 2$, we obtain the bound by induction.

$$|a_2| = \frac{|b_1|}{|[2]_{q(t)} - 1|} \le \frac{2}{[2]_{q(t)} - 1} = \frac{2}{1 + q(t) - 1} = \frac{2}{q(t)}.$$

Assume the bound holds for $k = 2, \dots, n-1$. Then:

$$|[\mathrm{n}]_{\mathrm{q(t)}} - 1||\mathrm{a_n}| \le \sum_{\mathrm{k=1}}^{\mathrm{n-1}} |\mathrm{a_{n-k}}||\mathrm{b_k}| \le 2\sum_{\mathrm{k=1}}^{\mathrm{n-1}} |\mathrm{a_{n-k}}|.$$

$$\Rightarrow |a_n| \le \prod_{k=0}^{n-2} \frac{[k+1]_{q(t)}+1}{[n-1]_{q(t)}!} \quad n > 1.$$

4. Applications

Example 4.1: Can we have $f(x)$ which is 1- times $h(t)$-differentiable but not 2- times?

Solution:

For any function $f: \mathbb{R} \to \mathbb{R}$ and any fixed $h(t) \neq 0$, the second h(t)-derivative exists everywhere. Therefore, there is no function $f(x)$ and $h(t)$ such that $f(x)$ is once $h(t)-$ differentiable but not twice $h(t) -$ differentiable at some point.

This is because,

$$D_{h(t)}f(x) = \frac{f(x+h(t)) - f(x)}{h(t)} \Rightarrow D_{h(t)}\left(D_{h(t)}f(x)\right) = \frac{f(x+2h(t)) - 2f(x+h(t)) + f(x)}{(h(t))^2}$$

It depends only on the values of $f(x)$ at x, $x + h(t)$, and $x + 2h(t)$. Since $f(x)$ is defined on $\mathbb{R}$, this is defined for all x and any fixed $h(t) \neq 0$.

Example 4.2: If $q(t) = 1 + t$ and $f(x) = |x|^3$ then find $q(t) -$ derivative of $f(x)$ for $x \neq 0$.

Solution:

Step I] Using the definition of $q(t) -$ derivative the first $(1 + t) -$ derivative of $f(x) = |x|^3$ is given by;

Case I] For $x \neq 0$

- If $x > 0$, $f(x) = x^3$, so:

$$D_{q(t)}^1 f(x) = \frac{f(q(t)x) - f(x)}{(q(t) - 1)x} = \frac{(q(t)x)^3 - x^3}{(q(t) - 1)x} = x^2(q(t)^2 + q(t) + 1).$$

With $q(t) = (1 + t)$

One can verify that as $t \longrightarrow 0 \Rightarrow D_{q(t)}^1 f(x) \longrightarrow \mathrm{D}^1 f(x) = 3x^2$

- If $x < 0$, $f(x) = -x^3$, so:

$$D_{q(t)}^{1}f(x)=\frac{f(q(t)x)-f(x)}{(q(t)-1)x}=\frac{-(q(t)x)^3-(-x^3)}{(q(t)-1)x}=-x^2(q(t)^2+q(t)+1).$$
With $q(t)=(1+t)$
One can verify that as $\mathrm{t}\longrightarrow 0 \Longrightarrow\ D_{q(t)}^{1}f(x)\longrightarrow\ \mathrm{D}^1 f(x)=-3x^2$

Step II] Using the definition of $q(t)-$ derivative the second $(1+t)-$ derivative of $f(x)=|x|^3$ is given by;

Case I] For $x\neq 0$

- If $x>0$, $D_{q(t)}f(x)=x^2(q(t)^2+q(t)+1)$,

$$\Longrightarrow D_{q(t)}^{2}f(x)=D_{q(t)}[x^2(q(t)^2+q(t)+1)]=(q(t)^2+q(t)+1)\cdot x(q(t)+1).$$
With $q(t)=(1+t)$
One can verify that as $\mathrm{t}\longrightarrow 0 \Longrightarrow\ D_{q(t)}^{2}f(x)\longrightarrow\ \mathrm{D}^2 f(x)=6x$

- If $x<0$, $D_{q(t)}f(x)=-x^2(q(t)^2+q(t)+1)$, so:

$$D_{q(t)}^{2}f(x)=D_{q(t)}[-x^2(q(t)^2+q(t)+1)]=-(q(t)^2+q(t)+1)\cdot x(q(t)+1).$$
With $q(t)=(1+t)$
One can verify that as $\mathrm{t}\longrightarrow 0 \Longrightarrow\ D_{q(t)}^{2}f(x)\longrightarrow\ \mathrm{D}^2 f(x)=-6x$

Step III] Using the definition of $q(t)-$ derivative the third $(1+t)-$ derivative of $f(x)=|x|^3$ is given by;

$$D_{q(t)}^{3}f(\mathrm{x})=\lim_{x\to 0}\frac{D_{q(t)}^{2}f(q(t)x)-D_{q(t)}^{2}f(x)}{(q(t)-1)x}.$$

- As $x\to 0^+$:
  $D_{q(t)}^{2}f(q(t)x)=(q(t)^2+q(t)+1)(q(t)+1)(q(t)x)$,

and $D_{q(t)}^{2}f(x)=(q(t)^2+q(t)+1)(q(t)+1)x$,

so:

$$\frac{[(q(t)^2+q(t)+1)(q(t)+1)(q(t)x)]-[(q(t)^2+q(t)+1)(q(t)+1)x]}{(q(t)-1)x}$$
$$=(q(t)^2+q(t)+1)(q(t)+1).$$
With $q(t)=(1+t)$
One can verify that as $\mathrm{t}\longrightarrow 0 \Longrightarrow\ D_{q(t)}^{3}f(x)\longrightarrow\ \mathrm{D}^3 f(x)=6$

- As $x\to 0^-$:

$$D_{q(t)}^{2}f(q(t)x)=-(q(t)^2+q(t)+1)(q(t)+1)(q(t)x),\quad D_{q(t)}^{2}f(x)$$
$$=-(q(t)^2+q(t)+1)(q(t)+1)x$$

so:

$$\frac{[-(q(t)^2+q(t)+1)(q(t)+1)(q(t)x)]-[-(q(t)^2+q(t)+1)(q(t)+1)x]}{(q(t)-1)x}$$
$$=-(q(t)^2+q(t)+1)(q(t)+1).$$
With $q(t)=(1+t)$

One can verify that as t $\longrightarrow 0 \Longrightarrow\ D^3_{q(t)}f(x) \longrightarrow\ \mathrm{D}^3 f(x) = -6$

Example 4.3: Solve the first order $q(t)$ linear differential equation $D_{q(t)}f(x) = 2f(x) + x$ with initial condition $f(0) = 1$.

Solution:

Step I] Using the definition of the $q(t)$ – derivative:

$$\frac{f(q(t)x) - f(x)}{(q(t)-1)x} = 2f(x) + x.$$

Multiplying both sides by $(q(t)-1)x$, with $q(\mathrm{t}) \neq 1$ and $\mathrm{q(t)} \to 1\ as\ t \to 0$

$$\Rightarrow f(q(t)x) - f(x) = (2f(x) + x)(q(t)-1)x$$

$$\Rightarrow f(q(t)x) = f(x)[1 + 2(q(t)-1)x] + (q(t)-1)x^2 \quad \text{--- (4.3.1)}$$

Step II] Assume a power series solution: $f(x) = \sum_{n=0}^{\infty} a_n\, x^n$, with $a_0 = f(0) = 1$.

$$\Rightarrow f(q(t)x) = \sum_{n=0}^{\infty} a_n\, q(t)^n x^n.$$

Substitute into the equation (4.3.1), it gives:

$$\sum_{n=0}^{\infty} a_n\, q(t)^n x^n = [1 + 2(q(t)-1)x] \sum_{n=0}^{\infty} a_n\, x^n + (q(t)-1)x^2$$

Now we expand the R. H. S. of the equation, to get.

$$\sum_{n=0}^{\infty} a_n\, q(t)^n x^n = \sum_{n=0}^{\infty} a_n\, x^n + 2(q(t)-1) \sum_{n=0}^{\infty} a_n\, x^{n+1} + (q(t)-1)x^2.$$

Shift the index in the second sum and Combining the terms:

$$\Rightarrow \sum_{n=0}^{\infty} a_n\, q(t)^n x^n = a_0 + \sum_{n=1}^{\infty} [a_n + 2(q(t)-1)a_{n-1}]\, x^n + (q(t)-1)x^2.$$

Step II] Equating the coefficients on the both sides we get;

$$a_0 = a_0 \quad \Rightarrow \quad a_0 = 1.$$

$$a_1 q(t) = a_1 + 2(q(t)-1)a_0 \Rightarrow a_1(q(t)-1) = 2(q(t)-1)$$

As $q(t) \neq 1 \Rightarrow a_1 = 2 = 2a_0$

$$a_2 q(t)^2 = a_2 + 2(q(t)-1)a_1 + (q(t)-1)$$

Substitute, $\mathrm{a}_1 = 2$ in the above equation and rearranging the terms:

$$a_2(q(t)^2 - 1) = 4(q(t)-1) + (q(t)-1) = 5(q(t)-1).$$

$$\Rightarrow a_2 = \frac{5(q(t)-1)}{(q(t)^2-1)},$$

As $q(t)^2 - 1 = (q(t)-1)(q(t)+1)$

$$\Rightarrow a_2 = \frac{5}{q(t)+1} = \frac{5}{[2]_{q(t)}},$$

- For $n \geq 3$

$$a_n q(t)^n = a_n + 2(q(t)-1)a_{n-1} \Rightarrow a_n(q(t)^n - 1) = 2(q(t)-1)a_{n-1}.$$

we know $q(t)^n - 1 = (q(t)-1)(1 + q(t) + \cdots + q(t)^{n-1})$,

$$\Rightarrow a_n = \frac{2}{1 + q(t) + \cdots + q(t)^{n-1}} a_{n-1} = \frac{2}{[n]_{q(t)}} a_{n-1}$$

Using this recurrence relation, it gives us:

$$a_n = a_2 \cdot \prod_{k=3}^{n} \frac{2}{[k]_{q(t)}} = \frac{5}{[2]_{q(t)}} \cdot \frac{2^{n-2}}{[3]_{q(t)}[4]_{q(t)} \cdots [n]_{q(t)}} = \frac{5 \cdot 2^{n-2}}{[2]_{q(t)}[3]_{q(t)} \cdots [n]_{q(t)}}$$

Step III] The unique analytic series solution to the initial value problem is:

$$f(x) = 1 + 2x + \sum_{n=2}^{\infty} \frac{5 \cdot 2^{n-2}}{[2]_{q(t)}[3]_{q(t)} \cdots [n]_{q(t)}} x^n - (4.3.2)$$

This power series converges for sufficiently small $|x|$, depending on the choice of $q(t)$.

The solution coincides with the actual series solution of the differential equation $f'(x) = 2f(x) + x$ with initial condition $f(0) = 1$ given by;

$$f(x) = 1 + 2x + \frac{5}{2}x^2 + \frac{5}{3}x^3 + \frac{5}{6}x^4 + \frac{1}{3}x^5 + \cdots - (4.3.3)$$

Where $q(t) \to 1 \; as \; t \to 0$

The analytic solution of the given problem under usual derivative is $\mathrm{f(x)} = \frac{5}{4}\mathrm{e}^{2\mathrm{x}} - \frac{\mathrm{x}}{2} - \frac{1}{4}$ with the given initial condition $\mathrm{f(0)} = 1$. $-$ (4.3.4)

For the equation (4.3.2)

$f(x) = 1 + 2x + \frac{5}{[2]_{q(t)}}x^2 + \frac{10}{[2]_{q(t)}[3]_{q(t)}}x^3 + \frac{20}{[2]_{q(t)}[3]_{q(t)}[4]_{q(t)}}x^4 + \frac{40}{[2]_{q(t)}[3]_{q(t)}[4]_{q(t)}[5]_{q(t)}}x^5$ at $x = 1$ and in series solution for the first 6 terms with $q(t) = 1 + t$ and $t = 0.001$

$At \; t = 0.001 \Rightarrow$

$$f(1) = 1 + 2 + \frac{5}{2.001} + \frac{10}{(2.001)(3.003001)} + \frac{20}{(2.001)(3.003001)(4.006004001)} + \frac{40}{(2.001)(3.003001)(4.006004001)(5.01001005001)}$$

≈ 8.325428

For the equation (4.3.3)

$f(x) = 1 + 2x + \frac{5}{2}x^2 + \frac{5}{3}x^3 + \frac{5}{6}x^4 + \frac{1}{3}x^5$ at at $x = 1$ and in series solution for the first 6 terms the approximate solution is given by $f(1) = 1 + 2 + \frac{5}{2} + \frac{5}{3} + \frac{5}{6} + \frac{1}{3} \approx 8.33333333$

For the equation (4.3.4)

Using the series solution of the given first order $q(t)$ - linear differential equation is given by

$$f(x) = \frac{5}{4}e^{2x} - \frac{x}{2} - \frac{1}{4}$$

Whose approximate value at $x = 1$ is given by

$$f(1) = \frac{5}{4}e^2 - \frac{1}{2} - \frac{1}{4} \approx 8.48632012$$

One can check that ads $\mathrm{t} \to 0$ the as $q(t) -$ Series Solution converges to the original solution.

Moreover, Geometrically, this operator captures the rate of change of f along a non-uniform scaling of the domain, determined by q(t). The solution f(x) can be interpreted as a curve whose tangent (in the q(t)-sense) at each point is proportional to its value plus a linear term. The initial condition fixes the curve's starting point. The geometry is enriched by the fact that the scaling factor q(t) introduces a deformation of the classical derivative, leading to solutions that may exhibit self-similar or fractal-like properties, depending on the choice of q(t). This connects to the broader geometric function theory where such equations describe mappings with specific distortion properties.

Example 4.4: Solve the first order $h(t)$ linear differential equation $D_{h(t)}f(x) = 2f(x) + x$ with initial condition $f(0) = 1$.

Solution:

Step I] Substitute the definition into the given equation, it gives us:

$$\frac{f(x + h(t)) - f(x)}{h(t)} = 2f(x) + x.$$

which gives;

$$f(x + h(t)) = (1 + 2h(t))f(x) + h(t)x.$$

Step II] Thereby framing the problem within a discrete dynamical system where the step size $h(t)$ governs the evolution of $f(x)$. This perspective reveals the inherent discrete structure underlying $h(t)$-calculus.

Thus we consider this problem in discrete frame that as it can be thought as of linear non – homogeneous recurrence relation**.**

Let $x = nh(t)$ and define $f_n = f(nh(t)) \Longrightarrow f_{n+1} = f\big((n+1)h(t)\big) + h(t)nh(t)$

Then the recurrence becomes:

$$\Longrightarrow f((n+1)h(t)) = (1 + 2h(t))f(nh(t)) + h(t)nh(t)$$
$$\Longrightarrow f_{n+1} = (1 + 2h(t))f_n + h(t)^2 n, \quad with\, f_0 = 1.$$

This is a linear non – homogeneous recurrence.

Step III] Homogeneous Solution

The homogeneous equation is:

$$f_{n+1} - (1 + 2h(t))f_n = 0,$$

with solution:

$$f_n^{(h)} = C(1 + 2h(t))^n$$

Step IV] Particular Solution

Assume that the particular solution is of the form:

$$f_n^{(p)} = An + B.$$

Substitute into the recurrence:

$$A(n+1) + B - (1 + 2h(t))(An + B) = h(t)^2 n$$

Simplify the equation:

$$An + A + B - An - 2h(t)An - B - 2h(t)B = -2h(t)An + A - 2h(t)B = h(t)^2 n.$$

Equating the coefficients of both sides of the above equation:

- For n : $-2h(t)A = h(t)^2 \Rightarrow A = -\frac{h(t)}{2}$,
- Constant term is: $A - 2h(t)B = 0$
  $\Longrightarrow -\dfrac{h(t)}{2} - 2h(t)B = 0$
  $\Longrightarrow B = -\dfrac{1}{4}$

Thus, the particular solution is:

$$f_n^{(p)} = -\frac{h(t)}{2}n - \frac{1}{4}$$

Step V] General Solution

Combine homogeneous and particular solutions, we get:

$$f_n = C(1 + 2h(t))^n - \frac{h(t)}{2}n - \frac{1}{4}.$$

Using $f_0 = 1$ gives, $C - \frac{1}{4} = 1 \Rightarrow \mathrm{C} = \frac{5}{4}$.

$$\Longrightarrow f_n = \tfrac{5}{4}(1 + 2h(t))^n - \tfrac{h(t)}{2}n - \tfrac{1}{4}.$$

Since, $x = nh(t)$, we have $n = \frac{x}{h(t)}$ as $h(t) \neq 0$

Substituting back, it gives us

$$f(x) = \frac{5}{4}(1 + 2h(t))^{x/h(t)} - \frac{x}{2} - \frac{1}{4}.$$

Step VI] Verification of Solution

$$f(x + h(t)) = \frac{5}{4}(1 + 2h(t))^{\frac{x+h(t)}{h(t)}} - \frac{x + h(t)}{2} - \frac{1}{4}$$
$$= \tfrac{5}{4}(1 + 2h(t))^{\frac{x}{h(t)}+1} - \tfrac{x}{2} - \tfrac{h(t)}{2} - \tfrac{1}{4}$$
$$\Longrightarrow f(x + h(t)) - f(x) = \frac{5}{4}(1 + 2h(t))^{\frac{x}{h(t)}}\left[\left(1 + 2h(t)\right) - 1\right] - \frac{h(t)}{2}$$

$$= \frac{5}{2} h(t)(1 + 2h(t))^{\frac{x}{h(t)}} - \frac{h(t)}{2}.$$
$$\Longrightarrow D_{h(t)} f(x) = \frac{5}{2}(1 + 2h(t))^{\frac{x}{h(t)}} - \frac{1}{2}$$

Now compute the right-hand side:

$$2f(x) + x = 2\left[\frac{5}{4}(1 + 2h(t))^{\frac{x}{h(t)}} - \frac{x}{2} - \frac{1}{4}\right] + x$$
$$= \frac{5}{2}(1 + 2h(t))^{x/h(t)} - x - \frac{1}{2} + x$$
$$= \frac{5}{2}(1 + 2h(t))^{x/h(t)} - \frac{1}{2}.$$

The two sides match, and $f(0) = 1$, so the solution is correct.

Such a solution using $h(t)$- derivative provides a discrete analogue of continuous differential equation and can be thought as foundation for numerical method and finite difference scheme. This is because, The $h(t)$-derivative is fundamentally a finite difference operator. The natural solution lives on the grid points $\{x_0, x_1, x_2, \dots\}$ generated by the recurrence $x_{n+1} = x_n + h(x_n)$.

Also, converting back to $f(x)$ for arbitrary $'x'$ requires inverting the mapping $x \mapsto x + h(x)$, which is generally impossible to express in elementary functions unless $h(x)$ has a very specific form.

Moreover, Geometrically this operator measures the rate of change of f(x) over intervals determined by h(t), capturing discrete or continuous deformations of the classical derivative. The solution f(x) describes a curve whose h(t)-incremental behavior at each point is proportional to its value plus a linear term. The initial condition anchors the curve at the origin.

The geometry here involves a deformation of the tangent concept, where the step size varies with position, leading to solutions that may exhibit properties of functions defined on non-uniform lattices or grids. This connects to the broader geometric function theory where such equations describe mappings with prescribed finite-difference characteristics.

5. Conclusion

This paper advances the theoretical framework of $q(t)$- and $h(t)$-calculus by establishing fundamental connections between these two formulations and their applications across multiple mathematical domains. We successfully introduced new classes of $q(t)$-special functions, including exponential, Gamma, Beta, Hypergeometric functions, and Hermite polynomials, while extending geometric function theory through $q(t)$-starlike and convex functions.

The key achievement lies in deriving the precise relationship between $h(t)$- and $q(t)$-derivatives and demonstrating that these calculi are not disjoint but complementary frameworks. The fundamental theorem of $h(t)$-calculus and orthogonality properties of $q(\tau)$-Hermite polynomials further validate the mathematical consistency of our approach.

Future Work:

The unified $q(t)$-$h(t)$ framework opens several promising research directions:

1. **Fractional Calculus**: Extending $q(t)$- and $h(t)$-operators to fractional orders could yield new fractional differential equations with applications in anomalous diffusion and viscoelasticity.

2. **Mathematical Physics**: The $q(t)$-hypergeometric series and Hermite polynomials may find applications in quantum mechanics, particularly in solving Schrödinger equations with position-dependent mass and in orthogonal polynomial theory.
3. **Complex Analysis**: Investigating $q(t)$-analytic functions and their mapping properties could lead to new univalence criteria and coefficient estimates in geometric function theory.
4. **Numerical Methods**: The $h(t)$-calculus fundamental theorem provides a basis for developing novel numerical integration schemes for non-uniform grids and adaptive algorithms.
5. **Special Function Theory**: Further exploration of $q(t)$-analogues of classical orthogonal polynomials and their generating functions may reveal new transformation formulas and recurrence relations with applications in approximation theory.

Acknowledgments: The authors express their sincere gratitude to our research center for providing the necessary resources and support for this research. Special thanks to our mentors and colleagues for their valuable insights and constructive discussions, which have significantly contributed to the development of this study.

Declaration: The authors confirm that the work presented in this paper is original and has not been submitted elsewhere for publication.

Data Availablity: All data available in manuscript.

Conflicts of Intrest: On behalf of all authors, the corresponding author states that there is no conflict of interest.

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