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# A Brief Review of The Development of The Theory of Dynamical Systems

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**Abstract:**

In this paper, we present a brief analysis of the evolution of theories of dynamical systems applied in various mathematical Models. In this study, the contributions of some notable mathematicians have been extensively discussed in the context of multiple applications of mathematical theories to dynamical systems. The investigation of the stability of periodic solutions, the stability of equilibrium, and Poincaré's periodicity has also been discussed in this paper. Moreover, an extensive study has been carried out on a brief application of the *Poincaré Maps*, the *Last Geometric Theorem,* the *Restricted Three-Body Problem,* the *Generalized Hopf bifurcation,* and the *Van der Pol and Lienard equation* for various mathematical models applied in dynamical systems.

**Keywords:** Dynamical Systems; Poincaré's contribution; Mathematical Modeling and Analysis; Aleksandr Lyapunov contribution; David Birkhoff Contribution.

## 1. Introduction

Poincaré developed some mathematical models for dynamical systems that could not be spread in a cumulative, linear fashion. For the first time, the works of Poincaré, Levi-Civita, and then Birkhoff introduced various concepts of stability, particularly in Birkhoff's work. Thus, it became possible for researchers to study the dynamical systems that evolve as a novel mathematical theory, particularly as a theory of mathematical modeling associated with qualitative nature in connection with stability analysis. Birkhoff highlighted many of Poincaré's works and challenged his conceptions in the stability analysis. In 1901, Birkhoff cited some publications by Levi-Civita, specifically an article that emphasizes the need for a more appropriate stability model in a qualitative approach. However, in this paper, we highlighted a full-length discussion of the various theories and applications of mathematical models based on the dynamical systems employed by some of the greatest mathematicians in the history of mathematics. The root of the development of the mathematical theory of dynamical systems can be traced back to Poincaré's research work based on differential equations and celestial mechanics from the 1880s. His research work can be classified mainly into four areas: (i) the qualitative theory, (ii) trajectories of global stability represented by such differential equations, (iii) the study of bifurcation of such dynamical systems, and (iv) probabilistic study of such systems with the help of Ergodic theory. We now examine these four areas one by one to understand the chronological development of the theories on dynamical systems contributed by Poincaré. Some of the great names of Mathematics, such as Cauchy, Jacobi, Lagrange, Laplace, d'Alembert, Clairaut, and Euler, developed the analysis of the Three-Body problem. During the 19th century, significant advancements in Newtonian mechanics were made to discuss the analytic structures of solutions obtained from the derivations of the equations of motion [3]. However, we have also seen that Pierre-Simon Laplace developed the perturbation method, which was first introduced by Leonhard Euler, to help him obtain accurate calculations of planetary orbits' distances as a power series in celestial mechanics. In 1846, Urbain Leverrier and John C. Adams developed computations of dynamical systems based on Neptune, which yielded potentially interesting results. The law of gravitation acting upon three masses- especially the Sun, the Earth, and the Moon- gave rise to a system of differential equations for which no explicit expression of the solution valid for all time could be found. However, in rational mechanics, one mostly tries to find a local trajectory, that is, solve a system of differential equations with given initial conditions without paying much attention to global behaviors. Mathematicians initially were able to yield a local solution, but obtaining a global solution was still a matter of considerable concern. For such a system, solutions were expressed in terms of power series.

## 2. Poincaré's Contribution to Dynamical Systems

In the late nineteenth century, Poincaré introduced methods based on topology, probability, and geometry to explain the qualitative and complex behavior of solutions to various nonlinear problems. Poincaré also analyzed this method to gain a better understanding of short-term motions, utilizing divergent series to aid astronomers. The major works by Poincaré on the Three-Body problem can be found in the memoire "Sur le probléme des trios corps et les équations de la dynamique" [1], for which he won the prize in 1889 on the 60$^{th}$ birthday of the king of Sweden. Several studies on the Three-Body problems have been observed in "*les méthodes Nouvelles de la mécanique celeste*" [1-2]. These memories, in vast part, discuss the previous research works of Poincaré and their further developments in the three-body problem. These research works are the significant sources of growth in modern dynamical system theory, including standard forms, exponents, invariant manifolds, divergence of perturbation series, variational equations, integral invariants, homoclinic and heteroclinic solutions, analytic non-integrability, the recurrence theorem, return maps, and surfaces of section. These theories contribute to the development of various fields, including symbolic dynamics, bifurcation theory, singularity theory, ergodic theory of invariant measures, the theory of K.A.M., the theory of weak and diffusion processes, symplectic geometry, and the development of theories related to computer experiments. In short, all new ideas evolving in Dynamical systems theory have roots in all these works of Poincaré. On the other hand, Poincaré included much literature in his book series "*Sur les courbes définies par une équation différentielle*" on the geometric and qualitative analysis of various mathematical models in the dynamical systems for the better understanding of the local and global phase portraits of solutions in the study of various differential equations, which are not easily solvable. Poincaré's work on differential equations, whether on Celestial mechanics or rotating fluid trajectories theory, or even his works on topology, always emphasized the global behavior of such differential equations. He introduced qualitative and geometric analysis in "Sur les courbes définies par une équation différentielle" to discuss general solutions and global behaviour in dynamical systems. Poincaré began by classifying two-dimensional solutions based on the singular points. He developed a concept based on the curve's index, which yielded the first local qualitative result through topographical comparison. He presented the idea of a transverse arc in his discussion of phase space, which does not terminate at a single point, wraps itself asymptotically around limit cycles, with some of them being periodic (i.e., limit cycles). Thus, a relatively accurate understanding of trajectories can be obtained by starting with behavior related to singular points, limit cycles, and transverse arcs. Furthermore, we should stress that Poincaré himself would gradually develop the qualitative theory into a completely new field of mathematics by using his geometric intuition, which would be especially helpful for the transition from local to global knowledge. Poincaré introduced the transverse arc by studying the phase space. However, this theorem was not enough to describe the global behaviors of the higher-dimensional system. Anyway, Poincaré's works on qualitative theory opened up a novel area of mathematics that led to the study of local to global knowledge.

Poincaré developed a qualitative theory to solve and analyze differential equations regarding celestial mechanics and the stability theorem for the solar system. He discussed these problems in his famous essay [4] and in his monumental treatise [5], where he developed a transverse section method and used a first-return map (or Poincaré map) to examine the behaviors within the neighborhood of periodic planner solutions. The Poincaré map proved to be a crucial component of the problem of lower-dimensional space, and the first discrete recurrence emerged in the theory of dynamical systems, where the study of trajectories is reduced to a point sequence in the normal plane, where time no longer varies continuously and is represented by integers. Historically, Lagrange was the first mathematician to raise the question of the stability of equilibrium and to formulate the criterion for verifying the stability of a system. According to Lagrange, "*an equilibrium position is stable if the system, if the equilibrium position is disturbed by a small amount, the system tends to that position by itself*." His proof was based on linearization, which is termed the "*Lagrange-Dirichlet theorem*". Lejeune Dirichlet [26] modified the result to understand that the concept of stability gained new meaning. The rigid periodicity of trajectories was what Lagrange meant by stability. Poisson had expanded it to include the scenario in which trajectories circled back indefinitely in their vicinities, rather than exactly to their initial positions. Poincaré elaborated the stability of solutions based on the trajectories, taking into account the departure of a trajectory from those that were initially near it. On the other hand, according to their characteristic exponents, both "stable" and "unstable" solutions are distinguished in the model. However, the results can be obtained from the Poincaré map, in which the phenomenon of transverse section contraction and dilation can be observed. Lyapunov and the Gorki School further developed the concept of global stability, which we will explore in our following sections. Poincaré developed the stability theorem, which was first published in 1885, a time when mathematicians were actively seeking connections between solutions of systems of differential equations and problems in celestial mechanics [27].

## 3. Mathematical Modeling and Analysis

**Poincaré stability of periodic solutions:** In the case of stable periodic solutions $\psi_i(t)$, all the characteristic exponents $\alpha_i = a_i + ib_i$ are purely imaginary numbers. So, it guarantees that all $\epsilon_i$ remain finite as $\epsilon_i = (\cos(b_i t) + isin(b_i t))S_{i,k}$ , where $S_{i,k}$ are periodic functions.

**Poincaré stability of equilibrium:** Let us assume a system that consists of $n$ quantities such as $x_1, x_2, \dots \dots, x_n$ and consider a force function $F(x_1, x_2, \dots \dots, x_n)$. A stable equilibrium position is obtained when the function attains its maximum, and its derivatives vanish to zero. We shall now mention three definitions as proposed by Poincaré:

**Poincaré stability on Poisson periodicity:** In linear systems, Poincaré defined the stability of solutions of a mathematical model governed by a set of differential equations in the neighbourhood of $x_i = \psi_i(t)$ i.e., periodic solution as follows:

$$\frac{d\epsilon_i}{dt} = \sum_{j=1,n} \frac{\partial \mathrm{X_i}}{\partial x_j} \epsilon_j \quad ,$$

where the term $\psi_{i(t)} + \epsilon_i(t)$ are considered as the perturbations of $x_i = \psi_i(t)$. The terms $\alpha_i$ in $\epsilon_1 = e^{\alpha_1 t} S_1, k, \ldots, \epsilon_n = e^{\alpha_n t} S_{n,k}$ are termed as "characteristic exponents".

In the study of stability, we will discuss the historical development of Birkhoff in the following sections. Studying stability theory, Poincaré developed two key concepts: asymptotic and doubly asymptotic solutions, also referred to as elliptic and hyperbolic solutions. Poincaré's works yield some beautiful implications. He also discovered that the family of trajectories of the periodic solutions has extensive periods, which produce islets and nodes, creating complex structures with regular and perturbed areas that repeat themselves in a minimal amount. Poincaré [33] proposed a new type of trajectory, which he called homoclinic, and it was very complicated for him to draw at that time. It took mathematicians years to draw such complex trajectories. Poincaré reduced the famous theorem, i.e., "*Theorem of Geometry*". However, Birkhoff proved Poincaré's "*Last Theorem*" [35, 36].

Poincaré mobilized the concept of probability, which is essential to understanding chaotic phenomena in dynamical systems. However, in the case of stability analysis of trajectories, Poincaré understood that there might be an infinite number of unstable solutions as per Poisson's prediction. In the dynamical system, the application of Probability theory and the Maxwell–Boltzmann postulate has been observed in the advancements of ergodic theory and the kinetic theory of gases [39, 40]. Thus, we have discussed the four major themes in the above. Based on the above discussion, it can be concluded that the study of dynamical systems and some of the theorems related to it was initially initiated by Poincaré's various works on celestial mechanics, which involved finding solutions to differential equations connected with those theories. He was the key person in coining the theory of Chaos. It is said that 'Chaos was discovered by Poincaré [37]. In 1908, French mathematician and physicist also claimed, "Small differences in the initial conditions may produce very great ones in the final phenomena. Prediction becomes impossible." [38]. This is the reason many mathematicians also term Poincaré a "Chaologist".

## 4. Contribution of Aleksandr Lyapunov to the Dynamical System:

For the first time in the 19th century, the Russian scientist N. E. Zhukovskii formulated and provided one of the most sophisticated mathematical definitions and theories of general stability regarding the stability of a dynamical system. In 1982, N. E. Zhukovskii (1847-1921), a Russian mathematician, introduced a concept of the stability of strong orbitals based on the reparameterization of the time variable, which agrees with Poincaré's stability theorem for equilibrium and solutions of a dynamical system. However, in 1892, another Russian mathematician A. M Lyapunov (1857-1918), who defended his PhD thesis "*A general task about the stability of motion*" where he introduced $n$ quantities i.e., $F_i$ functions of $k$ trajectories $f_i(t)$ of $q_i$ with initial condition $q_{j_0}$. The quantities $Q_j$ denotes the functions of perturbed trajectories for the perturbations $\epsilon_j$ of the initial positions and for the initial velocity $\epsilon_j{}'$ [41]. However, an equilibrium is also defined as a Lyapunov-stable system when there exists a delta-neighbourhood for a $\epsilon$-neighbourhood at the initial condition. In the original definition of Lyapunov, $L_i$ constructs $\epsilon$-*neighbourhood* but on the other hand, $E_j$ set up the $\delta$-neighbourhood. The original definition of Lyapunov was only restricted to the mechanical system with $k$ degrees of freedom for $n$ given functions $Q_j$ with $k$ position $q_j$. However, the modern definition of Lyapunov is not restricted to mechanical systems and is indeed applicable to any arbitrary dynamical system. The Lyapunov stability also implies that the neighbourhood solutions remain close to the state of equilibrium, which is similar to the idea of Lagrange's stability. Lyapunov established the stability theory in two ways: (1) Lyapunov's first indirect method, in which he proved the result based on linearization, and (2) the direct method of generalization, which is proved based on the Lagrange-Dirichlet stability theorem. The second method, also known as the direct method, can be applied to prove the stability of equilibrium by generalizing the concept of an energy function. However, in the 1960s, control theory emerged in Russia. Later, in 1907, a large group of researchers translated Lyapunov's works on the stability analysis of nonlinear systems. Apart from the Poincaré approach of stability of equilibrium, Lyapunov considered the Force function to develop a method that would solve such a problem without integrating. He defined stability by considering perturbed and non-perturbed motion. Now, we will discuss the stability of the equilibrium.

**Lyapunov's stability of equilibrium:**

Let us assume a system of differential equations as

$$\frac{dx_1}{dt} = X_1(x,t), \frac{dx_2}{dt} = X_2(x,t), \ldots, \frac{dx_n}{dt} = (x,t) \quad (1)$$

Now, we consider an equilibrium solution of the equation (1) be $(x_1, x_2, \ldots, x_n) = (0,0,0 \ldots,0)$. Thus, for each arbitrarily small $l \in \mathbb{Z}^+$, the solution of the equation (1) is stable. So, $\exists\ \epsilon \in \mathbb{Z}^+$ s.t. $||x_1(t)|| < l, \ldots, ||x_n(t)|| < l$, $\forall t \in \mathbb{Z}^+$, whenever $||x_1(t_0)|| < \epsilon, \ldots, ||x_n(t_0)|| < \epsilon$.

Thus, when the initial conditions of a trajectory approach the equilibrium state, the trajectory of the solution will be obtained within a small neighbourhood of the point $\epsilon$. Considering the transformation equation $(x_1, x_2, \ldots, x_n) \to (f_1(t), f_2(t), \ldots, f_n(t))$, we analyze the stability of a general solution $f_1(t), f_2(t), \ldots, f_n(t)$ that satisfy the differential equation

$$(X_1(0,t), X_2(0,t), \ldots, X_n(0,t)) = (0,0, \ldots,0).$$

However, when the solution of $\epsilon$-perturbation of its initial condition remains closed, the general solution of the above differential equation (1) becomes stable. Inspired by the works of Thomson and Tait on the linear approximations, Poincaré and Lyapunov developed their works on the stability of equilibrium in dynamical systems. Although Lagrange was the first to raise the question of the Stability of equilibrium, the most complete analysis of the stability of equilibrium was done by Thomson and Tait. They analyzed the stability of the Jacobi and Maclaurin ellipsoids based on the energy function. Poincaré defined the stability theorem based on the Energy function, and on the other hand, Lyapunov defined it considering space motion. Mawhin discussed the work of Lyapunov on the differential equations and theory of stability in [23], and he also compared Lyapunov's and Poincaré's different approaches

## 5. Application of Poincaré Maps

Let us consider the differential equation $\dot{y}(\mathrm{t}) = \mathrm{y}(\mathrm{t})$, with an assumed period of T = 1. The solution to this equation with an initial value $\mathrm{y}(0) = \mathrm{b}$ is given by $\mathrm{y}(\mathrm{t}) = \mathrm{b}\, e^t$. The Poincaré map is the function that provides the value of the solution at time T = 1, given by F(a) = be. This function is a linear map, and its graph is a simple straight line with slope (e). Fixed points of the Poincaré map satisfy the equation F (b) = b, which simplifies to e b = b. Since e is not equal to unity, the only solution is b = 0. Thus, the only fixed point of the Poincaré map is b = 0. This result is expected because fixed points of the Poincaré map correspond to periodic solutions of the differential equation. Among all solutions of the form y (t, b) = b $e^t$), only the trivial solution y(t) = 0 is periodic. The derivative of the Poincare map is $\dot{F}$(b) = e, which holds for any b. In particular, at the fixed-point b = 0, we have $\dot{F}(0)$ = e. Since e > 1, the fixed-point b = 0 is not asymptotically stable. This means that any small perturbation away from b = 0 will result in exponential divergence.Now, consider the nonlinear differential equation $\dot{y}(\mathrm{t}) = y^2(t)$, and again assume $T = 1$. The general solution of this equation is y(t) = $\frac{1}{t+k}$. For an initial condition y (0) = b, the solution takes the form y (t, b) = $\frac{b}{1-bt}$. The Poincaré map at T = 1 is given by P(b) = $\frac{b}{1-b}$. The Poincaré map is only defined for values of b where the solution exists for all t ∈ [0,1]. The solution becomes singular if $1 - bt = 0$, meaning the solution blows up before reaching t = 1. Therefore, the Poincaré map is only defined for b < 1.

Consider the linear differential equation $\dot{y}(\mathrm{t}) = -\mathrm{y}(\mathrm{t}) + 2\cos t$, which is periodic with period T = 2 $\pi$. It models a quantity y that undergoes exponential decay (due to -y) while being periodically replenished at a rate $2\cos t$. This is a linear differential equation that can be solved using an integrating factor $e^t$. The general solution is y(t) = $e^{-t}$ ($y_0$- 1) + $\cos t$ - $\sin t$. For huge t, the transient term $e^{-t}$ ($y_0$ - 1) vanishes, and the solution approximates y(t) to $\cos t$ - $\sin t$. To find the Poincaré map, we evaluate the solution at T = 2 $\pi$, and obtain P($y_0$) = $e^{-2\pi}$ ($y_0$ - 1) + 1, which simplifies to P($y_0$) = $e^{-2\pi}$ $y_0$ + 1- $e^{-2\pi}$. Fixed points satisfy P($y_0$) = $y_0$ Solving which gives the unique fixed point $y_0$ = 1. The derivative of the Poincaré map is $\dot{P}(\ y_0) = \ e^{-2\pi}$. Since 0 < $e^{-2\pi}$ < 1, the fixed point $y_0$ = 1 is asymptotically stable, meaning any small perturbation will decay over iterations of the Poincaré map, tending toward this steady-state value.

Again, we consider a 3D nonlinear system (Rossler attractor) as given below.

$$\dot{x}(\mathrm{t}) = -(\mathrm{y}(\mathrm{t}) + \mathrm{z}(\mathrm{t}),$$
$$\dot{y}(\mathrm{t}) = \mathrm{x}(\mathrm{t}) + \mathrm{p}\mathrm{y}(\mathrm{t}),$$
$$\dot{z}(\mathrm{t}) = \mathrm{q} + \mathrm{z}(\mathrm{t})(\mathrm{x}(\mathrm{t}) - \mathrm{r},$$

where p, q, and r are constants.

The Rossler attractor, defined by the chaotic behavior, can be studied with a Poincaré section by taking the extremum of the time series $x(t)$. By monitoring the sequence of maxima $x_{max}(k)$, crossings with the plane $y = -z$ are observed where $\dot{x}$(t) = 0. This method presents a one-dimensional map that describes the system's behavior at extremum points. A Poincaré section is characteristic of the intricate relationship between chaos and order in nonlinear dynamical systems.

Poincaré analysed the bifurcation of various figures of equilibrium, which shows the possibility of finding infinitely many such equilibrium figures from an ellipsoid. On the other hand, Lyapunov investigated the formation of the perturbed figures that are close to the original one.

Although the works of Poincaré and Lyapunov shared similarities, their differing approaches led to distinct definitions of stability. Poincaré published an article on the equilibrium figures of a rotating fluid in “Sur les courbes” [27]. However, the definition of Poison’s stability concerns individual trajectories but not the other trajectories in the neighbourhood. This definition of stability by Poincaré is less restrictive than Lyapunov's definition [28, 29]. Lyapunov also used the stability principle of Thomson and Tait for obtaining the extended form of Lagrange’s theorem in the case of fluids to describe the ellipsoidal figures and their stabilities. The definitions of stability and energy equations introduce the novel concept of equilibrium figures, which illustrate the close connection between them [30, 31]. The generic deformations of the figures indicate that there may be a unique characteristic of both fluid and equilibrium figures due to the presence of infinitely small perturbations and thin prominences. Thus, to exclude them, it would be necessary to create hardly verifiable conditions on the nature of initial perturbations. In the next section, we examine the contributions of G.D. Birkhoff, who advanced the qualitative study of dynamical systems to a more sophisticated level. A flowchart for depicting the evolution of the theory of dynamical systems has been shown in Figure 1.

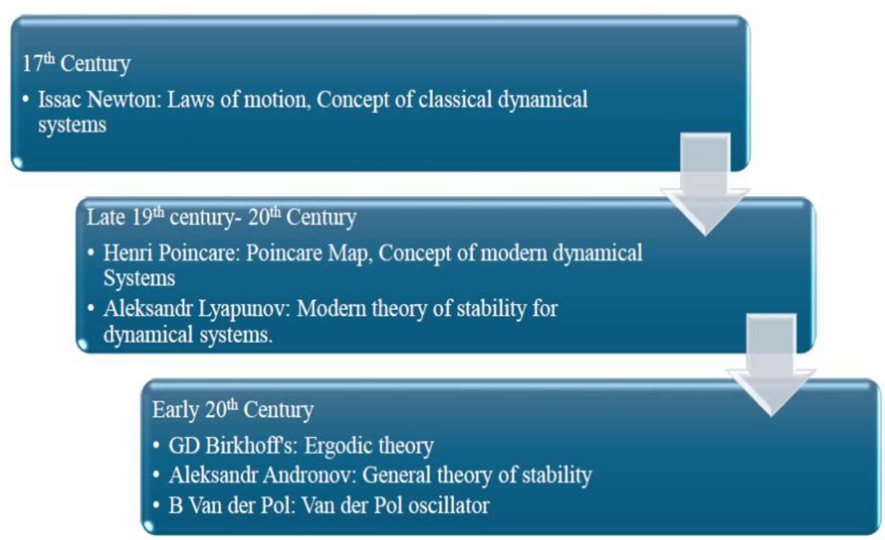


**Figure 1:** Evolution of the theory of dynamical systems

## 6. Birkhoff’s Contributions to Dynamical Systems:

In the history of mathematics, many mathematicians and astronomers developed the ideas of Poincaré in the field of dynamical systems. However, Birkhoff was quite different, as he extended the use of Poincaré topology in dynamical systems. He also provided his own point of view on the definition suggested by Poincaré, the three-body problem, i.e., stability as Poisson periodicity and one of the periodic trajectories. He also introduced a special solution, which is called "recurrent," which helps describe the entire set of trajectories [25]. Birkhoff used stability as a form of Poisson periodicity in characterizing recurrent movement and defined stability in terms of other criteria. Birkhoff defined "qualitative" stability as the set of trajectories that establish the stability of solutions, representing the behaviors of the solution in its neighborhood. According to Birkhoff, the definition that is based on the development of a series containing trigonometric terms only should be considered a kind of "formal" or "astronomical" stability. Thus, Poincaré again reinvestigated and analyzed the definition of stability of periodic solutions for dynamical systems. Birkhoff called this type of definition "perturbative" or "mathematical" stability. The "perturbative" stability has the same characteristics as defined by Lyapunov. Birkhoff developed the full-fledged theory of dynamical systems apart from the root of celestial mechanics as considered by Poincaré; instead, he took a topological approach. In 1912, after the demise of Poincaré, Birkhoff proved "*Poncaré's last geometric theorem*" [45, 46]. This theorem has some significant consequences in the development of the theory of dynamical systems.

It is observed that there is a close relation between Poincaré's theorem and the "*Restricted Three-Body problem*". In this theorem, Poincaré tried to prove the exact number of periodic motions in a dynamical system, whether it contains an infinite number of motions or not, for the same set of parametric values of the masses. Finally, in 1912, Birkhoff proved Poincaré's last theorem and published it in the Journal of the American Mathematical Society. Its first French translation appeared in 1913 [6]. Birkhoff gave the proof as follows: "*Let us suppose that a continuous one-to-one transformation T takes the ring [annulus] R formed by concentric $C_a$ and $C_b$ of radii a and b, respectively* $(a > b > 0)$ *into itself in such a way to advance the points of $C_a$ in a positive sense, and the points of $C_b$ in the negative sense, and at the same time to preserve areas. Then there are at least two invariant points*" [6]. In 1925, Birkhoff extended Poincaré's theorem and proved that in the case of annulus regions, the first and second invariant point possesses zero and two distinct invariant index points, respectively, for the arbitrary boundaries. These results were topological as the extension doesn't involve any invariant integral. This result implies the existence of infinitely many periodic motions near a stable motion in a dynamic system with two degrees of freedom. It asserts the existence of non-periodic motion, albeit limited to some periodic motion, specifically quasiperiodic motion. Three years after establishing this extension, Birkhoff started exploring the connections between the transformations of area-preserving and dynamical systems [8]. In this study, Birkhoff demonstrated that specific properties, based on area-preserving transformations, are essential for the motion of periodic solutions and the reverse part of a dynamical system. In one sentence, it means that there always exists a dynamical system correspondence to preserve the transformation. Birkhoff generalized this theorem into a higher-dimensional one in 1931 [9]. Due to these reasons, in 1924, Nikolai Krylov, a Russian mathematician, described Birkhoff as "*The Poincaré of America*".

In 1915, it was observed that Birkhoff carried out numerous works on the three-body problem [10]. However, the approaches to working on the three-body problems differ for both Poincaré and Birkhoff. Poincaré carried out extensive work on the concept of periodic solutions in the context of motion. However, Birkhoff extended his work through theoretical analysis and the application of topological sets in dynamical systems. He illustrated many problems of dynamical systems by considering the dependency of the Jacobian constant from a topological perspective. Birkhoff derived a set of novel equations that address regularity and the three-body system, based on the transformations of variables used in a dynamical system. In this study, he illustrated the types of motions that are expressed in terms of streamlines in a three-dimensional flow. The geometry of the states of motion is also represented with the help of non-singularity and one of the five values of the Jacobian constant. However, in the case of four-dimensional space, the streamlines also help to represent the states of motion without relying on the values of the Jacobian constant on the non-singular manifold. On the other hand, Poincaré stated that problems based on a three-dimensional flow may be represented depending on the transformation that occurs in a two-dimensional ring, provided the mass associated with these two bodies is very small [11]. In this investigation, Birkhoff also illustrated that the results obtained in Poincaré's transformation based on the existence of the amount of symmetric periodic motions as a characteristic and distribution property may be assumed as the multiplication of these two involutory transformations in the dynamical system.

Birkhoff also worked in the field of General Dynamical Systems and presented the result in a memoire in 1935 [12]. Later, he published two of his papers on restricted problems, taking an idea, known as the Poincaré section, from the memoire of Poincaré published in 1935. Poincaré also aimed at working on the analytic properties [13] and qualitative methods [14] to study the changes and surface sections of various kinds of relationships and motions that occur between them. In 1920, Birkhoff shared some ideas in his paper "Surface transformations and their dynamical applications" and extended his knowledge of the three-body problem, receiving the

Bôcher Prize [14]. In 1923, Birkhoff [15] discussed the general theory of dynamical systems and certain transformations along with their fixed points. However, a more detailed explanation of Poincaré stability can be found in the reference paper by Birkhoff [17]. In 1927, Birkhoff published a book entitled "*Dynamical Systems"*, which was also translated into Russian [19] in 1941. This book contains the translations of several research papers by Birkhoff. On the other hand, this book discusses the continuation of Poincaré's works on Celestial Mechanics, which is also a rare development of the theory of Dynamical systems in a topological environment [21].

## 7. Contribution of Aleksandr Aleksandrovich Andronov to the Dynamical System:

Andronov's self-acknowledgment of the theory of dynamical systems has two sources: (1) L. I Mandelstam, who was Andronov's mentor and a physicist who worked in optics, radiophysics, and in the theory of oscillations for the unification of the nature of the "Physics of oscillations", (2) the works of H. Poincaré. Andronov worked extensively on an engineering problem: "to take self-induction into account in the case of the electromagnetic switch," suggested by his mentor, Mandelstam. In van der Pol's relaxation oscillation, where the oscillator is dissipative, non-oscillatory external sources of energy sustain systemic vibrations, and Andronov [47] noticed that the motion in this oscillation is partially like Poincaré's limit cycles. Andronov developed a method named "storing method" to study and investigate the properties of stability of the periodic solutions in a dynamical system. Later, a novel method was developed based on Lyapunov's stability theory and Poincaré's small parameter method to obtain stability and periodic solutions in dynamical systems.

All these studies led Andronov and his collaborators to develop a novel theory of bifurcation, explicitly focusing on the double stability of the system, i.e., the system's ability to remain stable under initial variational conditions and parameter variations, as outlined in Lyapunov Theory. On the other hand, the stability of the periodic solution is apparent: "We have always to allow for the possibility of small variations of the form of the differential equations which describe a physical system". So, these studies led to a new notion in the theory of dynamical systems, which is the "coarse system" introduced by Andronov and Pontryagin (1937) in the lecture as "systemès grossiers". In his translation, Lefschetz calls this "coarse system" a "structurally stable system". Arnold [49] accentuated this new notion as a mathematically rigorous definition and its usefulness in modeling mathematical problems in engineering and physics.

Still, the search for a limit cycle regarding two critical questions concerning the stationary states involved only a few methods, such as Poincaré's index method and the Bendixson criterion [51]. It was essential to determine the qualitative structure of the orbits for the given system. In 1937, Anosov's school developed a coarse system of two dimensions and characterized bifurcations that may appear, as well as topological invariants. Within this coarse system, a stable limit cycle that oscillates was the only phenomenon observed. All these works contributed to solving numerous major engineering problems, including stabilization issues (i.e., the Mises-Vishegradsky problem) and various nonlinear oscillation problems. Oscillation problems related to diodes and magnetrons necessitate the statistical study of higher frequencies. Some researchers also conducted a survey of the effects of masers and lasers in dynamical systems in 1933 [51]. In 1937, the first edition of "*Theory of oscillations*" describes the mathematical proofs and examples of the birth of a limit cycle of the bifurcation in dynamical systems. In the aforementioned book, the recurrence relations of the differential equations in dynamical systems are also detailed. However, in the meantime, several researchers have applied bifurcation theory to investigate the various applications of dynamical systems for both second-order and higher-dimensional systems [52, 53, 54].

## 8. Contribution to the Dynamical System by Balthasar Van der Pol:

Van der Pol began working on a problem closely related to one that had already been solved by Andronov a few months prior. He described the equations of amplitude for oscillation for a current driven by the triodes. He also showed an example of dissipative equations that sustain spontaneous oscillations without forcing them, given by:

$$v'' - \epsilon(1 - v^2)v' + v = 0 .$$

In 1926, during the investigation of the behavior of larger values of ε, Van der Pol explained the phenomenon of frequency demultiplication in a dynamical system. However, with the help of electromagnetic theory, the graphical methods, and phase-space representations, the applications of the differential equations in general trajectories have been discussed. Later, numerous investigations were done into van der Pol equations, and various generalizations were made, among which the Lienard forced equation [55,56] is prominent, which is given by:

$$y + f(y)\dot{y} + y = p(t),$$

where $p(t)$ force term. However, to obtain the stable periodic solutions of a differential equation in a dynamical system, both engineers and mathematicians have studied the van der Pol and Lienard equations. Cartwright applied the topological methods, which were developed by Levinson, for discussing and explaining the Lienard equation as follows:

$$y - m(1 - y^2)\dot{y} + \ddot{y} = b\lambda \cos(\mu t + a).$$

On the other hand, in 1945, Littlewood *et al.* [57] discussed the existence of a periodic solution for the large values of $m$ and $b > \frac{2}{3}$. But at $< \frac{2}{3}$, they obtained a set $K_0$ of nonperiodic trajectories, which determines the measure of zero with two separated, bounded, and unbounded regions of the dynamical systems.

## 9. Conclusion

In this work, a brief analysis of the evolution and applications of mathematical theories and models has been conducted in various aspects of dynamical systems. The present study highlights the fundamental contributions of notable mathematicians, including Poincaré, Aleksandr Lyapunov, George David Birkhoff, and Balthasar Van der Pol, who utilized various applications of mathematical models in dynamical systems. Critical applications of these theories and models in dynamical systems, as used by famous mathematicians like Poincaré, Aleksandr Lyapunov, George David Birkhoff, and Balthasar Van der Pol, include the stability of periodic solutions, stability of equilibrium, and Poison's periodicity, which have also been included in this paper. While many applications of mathematical theories in dynamical systems can be observed, only a few have been broadly discussed in this work, such as the application of Poincaré Maps, the Last Geometric Theorem, the Restricted Three-Body Problem, the Generalized Hopf bifurcation, and the Van der Pol and Liénard equations. Additionally, many real-life problems based on mathematical models in dynamical systems can be explored and discussed. In conclusion, we extend our analysis to a theoretical approach that can be applied to computational mathematics across various scientific and engineering domains.

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