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Research Article

Oscillation Criteria of Fourth-Order Nonlinear Differential Equations of the Delay Type

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ABSTRACT

In this work, we establish new oscillation criteria for a class of fourth-order delay differential equations with noncanonical case. The paper presents certain oscillation criteria using a distinct comparison method approach using lower-order equations and generalized Riccati transformations. The capacity of the suggested approach to simplify the fourth-order problem by transforming it into first-order forms and enabling the use of proven oscillation theories is one of its main advantages. This method provides more effective and widely applicable outcomes in addition to extending current criteria to higher-order FDEs. The method's efficacy and wider applicability are confirmed by thorough comparisons with prior studies, which also show the viability and importance of our findings as an extension of earlier findings.

1. INTRODUCTION

An equation that describes the correlation between a function and its derivatives with respect to one or more independent variables is classified as a differential equation, where differentiation represents the instantaneous rate of change. Numerous scientific and biological phenomena in a variety of domains are continuously described and analyzed using this kind of equation. The 20th century witnessed a scientific revolution that accelerated progress across diverse disciplines, from genetics and biology to economics and social sciences. To model these complex phenomena, researchers have increasingly relied on various mathematical frameworks, employing ordinary,functional, and fractional differential equations to accurately describe and analyze real-world systems. Since conventional approaches to solving DEs frequently fail, this evolution and the requirement for such models necessitated the creation of new techniques. This progress catalyzed the establishment of a specialized field focused on the oscillatory behavior of solutions to DEs, one of the areas of qualitative theory that focuses on characterizing and evaluating DE solutions without determining their formulae [1–4].
One kind of functional differential equation (FDE) is a delay differential equation (DDE), where

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the derivative of an unknown function at any given point is characterized by the function's existing parameters of the function's values at earlier points in time. In many applications across the sciences, past dependency is a natural occurrence. It is simple to see that a variety of issues and events frequently have temporal memory and must be mathematically defined, accounting for their past or future conduct. As a result, DDEs are becoming more and more popular across all scientific disciplines, particularly in biology, economics, control engineering, and others [5–7]. Please refer to references [8–10] and the citations that go with them for a thorough review of PDEs and FDEs. Additionally, see references [11–13] and related publications for a detailed discussion on solving PDEs with sophisticated approaches including meshless techniques and deep learning.

Significant progress has been achieved in the study of oscillatory solutions, a key component of the qualitative theory of DEs, during the last ten years [14–16]. It was interested in examining the properties of oscillatory and non-oscillatory solutions. In the first quarter of the 20th century, Fite wrote a groundbreaking paper [17] that established the theory of oscillation of differential equations with oblique arguments. Researchers have extensively analyzed the oscillatory properties of diverse classes of equations, ranging from standard differential types to more complex functional forms. The interest in this topic is attested to by a number of monographs, such as Agarwal et al. [18], Gyori and Ladas [19] and Erbe et al. [20].

Our objective in this work is to derive sufficient conditions that ensure the oscillatory behavior of solutions for a class of fourth-order DDEs

$$\left[h\left(t\right)\left(w'''\left(t\right)\right)^{m}\right]'+\varsigma\left(t\right)f\left(w\left(\zeta\left(t\right)\right)\right)=0,\quad t\geq t_{0}. \tag{1}$$

The following presumptions are assumed to be true:

- (A_1) m is a quotient formed by odd positive integers;
- (A_2) $h\in C^{1}[t_{0},\infty)$, $h'\left(t\right)\geq 0$, $h\left(t\right)>0$, $\varsigma,\zeta\in C[t_{0},\infty)$, $f\in C\left(\mathbb{R},\mathbb{R}\right)$, ς is a positive constant, and the delay function $\zeta(t)$ satisfies $\zeta(t)\leq t$ with $\zeta(t)\to\infty$ as $t\to\infty$. Furthermore, the nonlinearity $f(u)$ is constrained by $f(u)/u^{m}\geq\beta$ for all non-zero u, where β is a positive constant.

We consider solutions $w(t)$ belonging to the space $C'''[T_w,\infty)$, such that the term $h(t)(w'''(t))^m$ is continuously differentiable. Our analysis focuses on non-trivial solutions—those that do not vanish identically for large t. A solution is classified as oscillatory if it possesses an infinite sequence of zeros tending to infinity; otherwise, it is nonoscillatory. If all solutions of Eq. (1-A) exhibit this behavior, the equation itself is termed oscillatory.

To underscore the distinctiveness of our results, we first explore several influential findings that have shaped the evolution of oscillation theory. A detailed comparison between these significant prior studies and our results is conducted to demonstrate the latter's scientific merit and innovative nature.

The oscillation of quasilinear delay differential equation is examined by Zhang, et al.[21] for

$$\left[h\left(t\right)\left(w'''\left(t\right)\right)^{m}\right]'+\varsigma\left(t\right)w^{m}\left(\zeta\left(t\right)\right)=0,$$

they using the comparison theorem, under

$$\int_{t_{0}}^{\infty}\frac{1}{h^{\frac{1}{m}}\left(t\right)}dt<\infty. \tag{2}$$

Grace, et al.[22] studied the oscillation behavior of the fourth-order DDE

$$\left[h\left(t\right)\left(w'\left(t\right)\right)^{m}\right]'''+\varsigma\left(t\right)f\left(w\left(g\left(t\right)\right)\right)=0,\qquad t\geq t_{0}.$$

They using the Riccati technique.
In their seminal work [23], Agarwal and his colleagues explored the conditions for oscillation in fourth-order DDEs

$$\left[r\left(t\right)\left(w''\left(t\right)\right)^{m}\right]'' + \varsigma\left(t\right)w^{m}\left(t\right) = 0, \qquad t \geq t_0,$$

under the conditions

$$\int_{t_0}^{\infty} \frac{1}{h^{\frac{1}{m}}\left(t\right)} dt = \infty. \tag{3}$$

In papers Agarwal et al. [24] and Zhang et al. [25], the oscillatory characteristics of the higher-order DDE are examined for

$$\left[h\left(t\right)\left(w^{(n-1)}\left(t\right)\right)^{m}\right]' + \varsigma\left(t\right)w^{m}\left(\zeta\left(t\right)\right) = 0, \qquad t \geq t_0.$$

Additionally, they used the comparison method with first-order equations under the condition (2) to achieve some better oscillation criterion.
Therefore, the behavior of fourth-order DE solutions is the main topic of this manuscript, with an emphasis on the characteristics of solutions for this equations. Next, we derive oscillation conditions that are more effective and adaptable than those employed in the noncanonical situation (2) of (1) using the well-known generalized Riccati transformations.

2. NEW OSCILLATION CONDITIONS

This section is devoted to establishing conditions that ensure the oscillation of solutions to Eq. (1-A). Our investigation of the asymptotic behavior of positive solutions reveals that they can be categorized into four fundamental scenarios:

$$\begin{aligned}
&\text{Case (1)}: \quad w^{(r)}\left(t\right) > 0 \quad \text{for} \quad r = 0, 1, 2, 3.\\
&\text{Case (2)}: \quad w^{(r)}\left(t\right) > 0 \quad \text{for} \quad r = 0, 1, 3 \text{ and } w''\left(t\right) < 0.\\
&\text{Case (3)}: \quad w^{(r)}\left(t\right) < 0 \quad \text{for} \quad r = 1, 3 \text{ and } w''\left(t\right) > 0.\\
&\text{Case (4)}: \quad w^{(r)}\left(t\right) > 0 \quad \text{for} \quad r = 0, 1, 2 \text{ and } w'''\left(t\right) < 0.
\end{aligned}$$

Utilizing the relationship defined in Eq. (1-A) and the constraints in (2-A), we observe the behavior of $(h(t)(w'''(t))^m)'$. We proceed by introducing some essential lemmas.

Lemma 1. *[26]Let $y \in C^n([t_0, \infty), \mathbb{R}^+)$ such that $y(t) > 0$ and $y^{(n-1)}(t)y^{(n)}(t) \leq 0$. Suppose $\lim_{t\to\infty} y(t) \neq 0$ and $y^{(n)}$ has a fixed sign without being identically zero. Then, for every $\varepsilon \in (0, 1)$, there exists $t_\varepsilon \geq t_0$ such that:*

$$y(t) \geq \frac{\varepsilon}{(n-1)!} t^{n-1} |y^{(n-1)}(t)|, \quad t \in [t_\varepsilon, \infty).$$

We are now ready to present and prove our main findings. To simplify the presentation, we define:

$$x(s) := \int_{t_0}^{\infty} \frac{1}{h(u)} du, \quad \xi'_+(t) := \max\{0, \xi'(t)\}, \quad \text{and} \quad \phi'_+(t) := \max\{0, \phi'(t)\}.$$

Lemma 2. *Suppose w is a solution of (1) that is eventually positive and satisfies $w^{(r)}(t) > 0$ for each $r \in \{1, 2, 3\}$. Given that ξ belongs to the class $C^1[t_0, \infty)$, it follows that:*

$$x'\left(t\right) \leq -\beta\varsigma\left(t\right)\left(\frac{\zeta^3\left(t\right)}{t^3}\right)^{m} \xi\left(t\right) + \frac{\xi'\left(t\right)}{\xi\left(t\right)} x\left(t\right) - \frac{m\varepsilon t^2}{2\left(h\left(t\right)\xi\left(t\right)\right)^{\frac{1}{m}}} x^{\frac{m+1}{m}}\left(t\right). \tag{4}$$

for all $t > t_1$, where t_1 large enough.

Proof. Let w be a solution of Eq. (1) that satisfies $w^{(r)}(t) > 0$ for $r = 0, 1, 2, 3$ for all sufficiently large t. Define

$$x(t) := \xi(t) \frac{h(t)(w''')^m(t)}{w^m(t)}. \tag{5}$$

Then $x(t) > 0$ for $t \geq t_1$ and

$$x'(t) = \xi'(t) \frac{h(t)(w''')^m(t)}{w^m(t)} + \xi(t) \frac{(h(w''')^m)'(t)}{w^m(t)} - m\xi(t) \frac{w^{m-1}(t)(w)'(t)h(t)(w''')^m(t)}{w^{2m}(t)}. \tag{6}$$

Applying Lemma (1), we obtain the following inequality for any $\varepsilon \in (0, 1)$ and all sufficiently large t:

$$w'(t) \geq \frac{\varepsilon}{2} t^2 w'''(t). \tag{7}$$

By this Lemma

$$\frac{w!}{t^r} w(t) - \frac{(r-1)!}{t^{r-1}} \frac{d}{dt} w(t) \geq 0,$$

for $w^{(k)} > 0$ and $w^{(r+1)} < 0$ for all $k = 0, 1, ..., r$. Thus, we find

$$w(t) \geq (t/3) w'(t),$$

and, hence

$$\frac{w(\zeta_i(t))}{w(t)} \geq \frac{\zeta^3(t)}{t^3}. \tag{8}$$

By (7) and (8), we obtain

$$x'(t) \leq \xi'(t) \frac{h(t)(w''')^m(t)}{w^m(t)} + \xi(t) \frac{(h(w''')^m)'(t)}{w^m(t)} - \frac{m\varepsilon}{2} t^2 \xi(t) \frac{w'''(t)h(t)(w''')^m(t)}{(w)^{m+1}(t)}. \tag{9}$$

In view of (1), we get

$$x'(t) \leq -\beta\varsigma(t) \left(\frac{\zeta^3(t)}{t^3}\right)^m \xi(t) + \frac{\xi'(t)}{\xi(t)} x(t) - \frac{m\varepsilon t^2}{2(h(t)\xi(t))^{\frac{1}{m}}} x^{\frac{m+1}{m}}(t).$$

The proof is complete.

Lemma 3. *If Case* (2) *hold and w is a solution of (1) that is eventually positive. If we have the function* $\phi \in C^1[t_0, \infty)$, *then*

$$\varkappa'(t) \leq -\phi(t) \int_t^\infty \left[\frac{1}{h(\vartheta)} \int_y^\infty \beta\varsigma(s) \left(\frac{\zeta^3(s)}{s^3}\right)^m ds\right]^{\frac{1}{m}} d\vartheta + \frac{(\phi'(t))^2}{4\phi(t)}. \tag{10}$$

Proof. Let Case (2) hold and w is an eventually positive solution of (1). Define

$$\varkappa(t) := \phi(t) \frac{w'(t)}{w(t)}, \ t \geq t_1. \tag{11}$$

Then $\varkappa(t) > 0$ for $t \geq t_1$ and

$$\varkappa'(t) = \phi'(t) \frac{w'(t)}{w(t)} - \phi(t) \frac{w''(t)w(t) - (w')^2(t)}{w^2(t)}, \tag{12}$$

Differentiating Eq. (11) yields the following expression for $\varkappa'(t)$:

$$\varkappa'(t) = \phi(t)\frac{w''(t)}{w(t)} + \frac{\phi'(t)}{\phi(t)}\varkappa(t) - \frac{\varkappa^2(t)}{\phi(t)}.$$

Next, by integrating the main equation (1) over the interval $[t, u]$, we obtain:

$$h(u)\left(w'''\right)^m(u) - h(t)\left(w'''\right)^m(t) + \int_t^u \beta_{\varsigma}(s)\, w^m(\zeta_i(s))\, ds \leq 0,$$

From (8), we obtain

$$h(u)\left(w'''\right)^m(u) - h(t)\left(w'''\right)^m(t) + w^m(t)\int_t^u \beta_{\varsigma}(s)\left(\frac{\zeta^3(s)}{s^3}\right)^m ds \leq 0.$$

Upon taking the limit as $u \to \infty$ in the previous expression, we derive the inequality:

$$w''(t) + w(t)\int_t^{\infty}\left[\frac{1}{h(\vartheta)}\int_y^{\infty}\beta_{\varsigma}(t)\left(\frac{\zeta^3(s)}{s^3}\right)^m ds\right]^{\frac{1}{m}} d\vartheta \leq 0. \tag{13}$$

Hence, by (13) in (12), we find

$$\begin{aligned}\varkappa'(t) \quad \leq \quad & -\phi(t)\int_t^{\infty}\left[\frac{1}{h(\vartheta)}\int_{\vartheta}^{\infty}\beta_{\varsigma}(t)\left(\frac{\zeta^3(s)}{s^3}\right)^m ds\right]^{\frac{1}{m}} d\vartheta \\ & + \frac{\phi'(t)}{\phi(t)}\varkappa(t) - \frac{\varkappa^2(t)}{\phi(t)}.\end{aligned}$$

Thus, we have

$$\varkappa'(t) \leq -\phi(t)\int_t^{\infty}\left[\frac{1}{h(\vartheta)}\int_y^{\infty}\beta_{\varsigma}(s)\left(\frac{\zeta^3(s)}{s^3}\right)^m ds\right]^{\frac{1}{m}} d\vartheta + \frac{(\phi'(t))^2}{4\phi(t)}.$$

The proof is complete.

Lemma 4. *If Case* (3) *hold and w be an eventually positive solution of (1), then*

$$\xi(t)\left(\int_t^{\infty}\int_u^{\infty} x(s)\, dsdu\right)^m \geq -1. \tag{14}$$

Proof. Let w be an eventually positive solution of (1) and assume Case (3) holds. Since $r(t)(w'''(t))^m$ is a nonincreasing function, it follows that:

$$\begin{aligned} h^{\frac{1}{m}}(s)\, w'''(s) \quad & \leq \quad h^{\frac{1}{m}}(t)\, w'''(t), \ s \geq t \geq t_1, \\ w'''(s) \quad & \leq \quad h^{\frac{1}{m}}(t)\, w'''(t)\, h^{\frac{-1}{m}}(s). \end{aligned}$$

Integrating the resulting inequality again from t to v yields

$$w''(t) - w''(v) \geq -h^{\frac{1}{m}}(t)\, w'''(t)\int_t^v h^{\frac{-1}{m}}(s)\, ds.$$

By taking the limit as $v \to \infty$, we derive the following inequality

$$w''(t) \geq -h^{\frac{1}{m}}(t)w'''(t)x(t). \tag{15}$$

Integrating from t to ∞, we get

$$-w'(t) \geq -h^{\frac{1}{m}}(t)\, w'''(t) \int_t^{\infty} x(s)\, ds. \tag{16}$$

Integrating again from t to ∞, we get

$$w'(t) \geq -h^{\frac{1}{m}}(t)\, w'''(t) \int_t^{\infty} \int_u^{\infty} x(s)\, dsdu. \tag{17}$$

We define

$$\xi(t) := \frac{h(t)\,(w''')^m(t)}{w^m(t)}. \tag{18}$$

By (18), we conclude that

$$\xi'(t) = \frac{(h\,(w''')^m)'(t)}{w^{\,m}(t)} - m\frac{(w)'(t)\,h(t)\,(w''')^m(t)}{w^{m+1}(t)}.$$

$$\xi'(t) \leq -\beta\varsigma(t)\left(\frac{w(\zeta(t))}{w(t)}\right)^m - m\frac{h^{\frac{m+1}{m}}(t)\,(w''')^{m+1}(t)}{w^{m+1}(t)} \int_t^{\infty} x(s)\, ds. \tag{19}$$

Hence, by (19) and (18), we obtain

$$\xi'(t) \leq -\beta\varsigma(t) - m\xi^{\frac{m+1}{m}}(t) \int_t^{\infty} x(s)\, ds,$$

from (6), we get

$$\xi(t)\left(\int_t^{\infty} \int_u^{\infty} x(s)\, dsdu\right)^m \geq -1.$$

The proof is complete.

Lemma 5. *Assume that w be an eventually positive solution of (1) and Case* (4) *hold, then*

$$\delta'(t) \leq -\beta\varsigma(t)\left(\frac{\varepsilon}{2}\zeta^2(t)\right)^m - m\frac{\delta^{\frac{m+1}{m}}(t)}{h^{\frac{1}{m}}(t)}. \tag{20}$$

Proof. Let w is an eventually positive solution of (1) and Case (4) hold. By Lemma (2), we have (6). On the other hand, by Lemma (1), we get

$$w(t) \geq \frac{\varepsilon}{2}t^{\,2}w''(t), \tag{21}$$

for every $\varepsilon \in (0,\ 1)$ and all sufficiently large t. We now define

$$\delta(t) := \frac{h(t)\,(w''')^m(t)}{(w'')^m(t)}. \tag{22}$$

Then $\delta(t) < 0$ for $t \geq t_1$ and, by (22), we see that

$$\begin{aligned}\delta'(t) &= -\beta\varsigma(t)\frac{w^m(\zeta(t))}{(w''(\zeta(t)))^m}\frac{(w''(\zeta(t)))^m}{(w'')^m(t)} - m\frac{\delta^{\frac{m+1}{m}}(t)}{h^{\frac{1}{m}}(t)}.\\ &= -\beta\varsigma(t)\left(\frac{w(\zeta(t))}{w''(\zeta(t))}\right)^m\frac{(w''(\zeta(t)))^m}{(w''(t))^m} - m\frac{\delta^{\frac{m+1}{m}}(t)}{h^{\frac{1}{m}}(t)}.\end{aligned}$$

By (21), we obtain

$$\delta'(t) \leq -\beta\varsigma(t)\left(\frac{\varepsilon}{2}\zeta^2(t)\right)^m - m\frac{\delta^{\frac{m+1}{m}}(t)}{h^{\frac{1}{m}}(t)}.$$

The proof is complete.

Theorem 1. *Let (2) hold. Assume that there exists a positive functions $\xi \in C^1[t_0, \infty)$ and $\phi \in C^1[t_0, \infty)$ such that*

$$\int_{t_0}^{\infty}\left[\beta\varsigma(s)\left(\frac{\zeta^3(s)}{s^3}\right)^m\xi(s) - \frac{2^m h(s)(\xi'(s))^{m+1}}{(m+1)^{m+1}\left(\varepsilon_1\xi(s)s^2\right)^m}\right]ds = \infty, \tag{23}$$

and

$$\int_{t_0}^{\infty}\left[\phi(s)\int_s^{\infty}\left[\frac{1}{h(\vartheta)}\int_{\delta}^{\infty}\beta\varsigma(\nu)^m d\nu\right]^{\frac{1}{m}}d\vartheta - \frac{(\phi'(s))^2}{4\phi(s)}\right]ds = \infty. \tag{24}$$

Also, if

$$\int_{t_0}^{\infty}\left[\beta\varsigma(s)\left(\int_s^{\infty}\int_u^{\infty}x(\chi)\,d\chi du\right)^m - \frac{m^{m+1}\int_s^{\infty}x(\chi)d\chi}{(m+1)^{m+1}\int_s^{\infty}\int_u^{\infty}x(\chi)d\chi du}\right]ds = \infty, \tag{25}$$

and

$$\int_{t_0}^{\infty}\left[\beta\varsigma(s)\left(\frac{\varepsilon_2}{2}\zeta^2(s)\right)^m x^m(s) - \frac{m^{m+1}}{(m+1)^{m+1}\ x(s)\,h^{\frac{1}{m}}(s)}\right]ds = \infty, \tag{26}$$

for some constant $\varepsilon_1, \varepsilon_2 \in (0,\ 1)$, then every solution of (1) is oscillatory.

Proof. Assume that (1) has a nonoscillatory solution w. Without loss of generality, we can assume that $w(t) > 0$. From Lemma (2), we have

$$x'(t) \leq -\beta\varsigma(t)\left(\frac{\zeta^3(t)}{t^3}\right)^m\xi(t) + \frac{\xi'(t)}{\xi(t)}x(t) - \frac{m\varepsilon t^2}{2\left(h(t)\xi(t)\right)^{\frac{1}{m}}}x^{\frac{m+1}{m}}(t).$$

Applying the inequality

$$Dy - Cy^{\frac{m+1}{m}} \leq \frac{m^m}{(m+1)^{m+1}}\frac{D^{m+1}}{C^m},\ C,\ D > 0, \tag{27}$$

such that

$$C := \frac{m\varepsilon t^2}{2\left(h(t)\xi(t)\right)^{\frac{1}{m}}},\ D := \frac{\xi'(t)}{\xi(t)},\ y := x(t).$$

So, we find

$$\frac{\xi'(t)}{\xi(t)}x(t) - \frac{m\varepsilon t^2}{2\left(h(t)\xi(t)\right)^{\frac{1}{m}}}x(t)^{\frac{m+1}{m}} \leq \frac{2^m}{(m+1)^{m+1}}\frac{h(t)(\xi'(t))^{m+1}}{(\varepsilon\xi(t)t^2)^m}. \tag{28}$$

Hence, we obtain

$$x'(t) \leq -\beta\varsigma(t)\left(\frac{\zeta^3(t)}{t^3}\right)^m \xi(t) + \frac{2^m}{(m+1)^{m+1}} \frac{h(t)(\xi'(t))^{m+1}}{(\varepsilon\xi(t)t^2)^m}.$$

Integrating from t_1 to t, we get

$$\int_t^t \left[-\beta\varsigma(s)\left(\frac{\zeta^3(s)}{s^3}\right)^m \xi(s) - \frac{2^m h(s)(\xi'(s))^{m+1}}{(m+1)^{m+1}(\varepsilon_1\xi(s)s^2)^m}\right] ds \leq x(t_1),$$

for any $\varepsilon \in (0,\ 1)$, and all sufficiently large t although this is in conflict with (23).
By Lemma (3), we have

$$\varkappa'(t) \leq -\phi(t)\int_t^\infty \left[\frac{1}{h(\vartheta)}\int_y^\infty \beta\varsigma(s)\left(\frac{\zeta^3(s)}{s^3}\right)^m ds\right]^{\frac{1}{m}} d\vartheta + \frac{(\phi'(t))^2}{4\phi(t)}.$$

Integrating from t_1 to t, we get

$$\int_{t_1}^t \left[\phi(s)\int_s^\infty \left[\frac{1}{h(\vartheta)}\int_\delta^\infty \beta\varsigma(v)\left(\frac{\zeta^3(v)}{v^3}\right)^m dv\right]^{\frac{1}{m}} d\vartheta - \frac{(\phi'(s))^2}{4\phi(s)}\right] ds \leq \varkappa(t_1).$$

which contradicts (24).
By Lemma (4), we have

$$\xi(t)\left(\int_t^\infty \int_u^\infty x(s)\,dsdu\right)^m \geq -1.$$

By multiplying (19) by $\left(\int_t^\infty \int_u^\infty x(s)\,dsdu\right)^m$ and integrating the resultant inequality from t_1 to t, we obtain

$$\begin{aligned}
&\left(\int_t^\infty \int_u^\infty x(s)\,dsdu\right)^m \xi(t) - \left(\int_{t_1}^\infty \int_u^\infty x(s)\,dsdu\right)^m \xi(t_1) \\
&+m\int_{t_1}^t \int_s^\infty x(\chi)\,d\chi\left(\int_s^\infty \int_u^\infty x(\chi)\,d\chi du\right)^{m-1} \xi(s)\,ds \\
&+\int_{t_1}^t \beta\varsigma(s)\left(\int_s^\infty \int_u^\infty x(\chi)\,d\chi du\right)^m ds
\end{aligned}$$

$$+m\int_{t_1}^t \xi^{\frac{m+1}{m}}(s)\left(\int_s^\infty \int_u^\infty x(\chi)\,d\chi du\right)^m \int_s^\infty x(\chi)\,d\chi ds \leq 0.$$

We set

$$D := \int_s^\infty x(\chi)\,d\chi\left(\int_s^\infty \int_u^\infty x(\chi)\,d\chi du\right)^{m-1},$$

$$C := \left(\int_s^\infty \int_u^\infty x(\chi)\,d\chi du\right)^m \int_s^\infty x(\chi)\,d\chi,\ y := -\xi(s).$$

Applying the inequality (27). we conclude that

$$\begin{aligned}
&\int_s^\infty x(\chi)\,d\chi\left(\int_s^\infty\int_u^\infty x(\chi)\,d\chi du\right)^{m-1}\xi(s)\\
&+\xi^{\frac{m+1}{m}}(s)\left(\int_s^\infty\int_u^\infty x(\chi)\,d\chi du\right)^{m}\int_s^\infty x(\chi)\,d\chi\\
\geq\ & \frac{m^m\int_s^\infty x(\chi)\,d\chi}{(m+1)^{m+1}\int_s^\infty\int_u^\infty x(\chi)\,d\chi du}.
\end{aligned}$$

Consequently, upon integrating both sides from t_1 to t and utilizing (14), it follows that

$$\begin{aligned}
&\int_{t_1}^t\left[\beta_\varsigma(s)\left(\int_s^\infty\int_u^\infty x(\chi)\,d\chi du\right)^m-\frac{m^{m+1}\int_s^\infty x(\chi)d\chi}{(m+1)^{m+1}\int_s^\infty\int_u^\infty x(\chi)d\chi du}\right]ds\\
\geq\ &\left(\int_{t_1}^\infty\int_u^\infty x(s)\,dsdu\right)^m\xi(t_1)+1,
\end{aligned}$$

which contradicts (25).
From Lemma (4), we have

$$\delta'(t)\leq-\beta_\varsigma(t)\left(\frac{\varepsilon}{2}\zeta^2(t)\right)^m-m\frac{\delta^{\frac{m+1}{m}}(t)}{h^{\frac{1}{m}}(t)}.$$

After weighting the inequality with the factor $x^m(t)$ and integrating over the interval $[t_1,t]$, we obtain the following estimate

$$\begin{aligned}
&x^m(t)\ \delta(t)-x^m(t_1)\,\delta(t_1)+m\int_{t_1}^t h^{\frac{-1}{m}}(s)\,x^{m-1}(s)\,\delta(s)\,ds\\
\leq\ &-\int_{t_1}^t\beta_\varsigma(s)\left(\frac{\varepsilon}{2}\zeta_i^2(s)\right)^m x^m(s)\,ds-m\int_{t_1}^t\frac{\delta^{\frac{m+1}{m}}(t)}{h^{\frac{1}{m}}(t)}x^m(s)\,ds.
\end{aligned}$$

We set

$$D:=h^{\frac{-1}{m}}(s)\,x^{m-1}(s),\ C:=\frac{x^m(s)}{h^{\frac{1}{m}}(t)},\ y:=-\delta(s).$$

Applying the inequality (27) and (15), for every $\varepsilon\in(0,\ 1)$, we findin

$$\int_{t_1}^t\left[\beta_\varsigma(s)\left(\frac{\varepsilon}{2}\zeta^2(s)\right)^m x^m(s)-\frac{m^{m+1}}{(m+1)^{m+1}x(s)h^{\frac{1}{m}}(s)}\right]ds\leq x^m(t_1)\,\delta(t_1)+1,$$

but this contradicts (26). Theorem (1) is proved.

3. EXAMPLE

To demonstrate our primary findings, we provide the following example in this section.

Example 1. *We have the equation*

$$\left(t^6\left(w'''(t)\right)\right)' + \eta t^2 w(t) = 0,\ t \geq 1, \tag{29}$$

where $\eta > 0$ is a constant. Let

$$\begin{aligned} m &= 1,\ h(t) = t^6 > 0,\ h'(t) = 6t^5 \geq 0,\ h \in C^1[t_0, \infty), \\ \varsigma(t) &= \eta t^2 > 0,\ \varsigma \in C[t_0, \infty), \\ \zeta(t) &= t,\ \lim_{t\to\infty} t = \infty,\ \zeta(t) \in C[t_0, \infty). \end{aligned}$$

Thus, we get

$$x(t) = \frac{1}{5t^5},\ \int_s^\infty x(\chi)\,d\chi = \frac{1}{20s^4},\ \int_s^\infty \int_u^\infty x(\chi)\,d\chi du = \frac{1}{60s^3}.$$

If we set $\xi(t) = \phi(t) = \beta = 1$, then by applying this example to conditions (23), (24), (25) and (26), we find that

$$\begin{aligned} &\int_{t_0}^\infty \left[\beta\varsigma(s)\left(\tfrac{\zeta^3(s)}{s^3}\right)^m \xi(s) - \tfrac{2^m h(s)(\xi'(s))^{m+1}}{(m+1)^{m+1}\left(\varepsilon_1\xi(s)s^2\right)^m}\right] ds, \\ = &\int_{t_0}^\infty \eta s^2 ds \\ = &\infty, \end{aligned}$$

and

$$\begin{aligned} &\int_{t_0}^\infty \left[\phi(s)\int_s^\infty \left[\tfrac{1}{h(\vartheta)}\int_\delta^\infty \beta\varsigma(v)^m\,dv\right]^{\frac{1}{m}} d\vartheta - \tfrac{(\phi'(s))^2}{4\phi(s)}\right] ds \\ = &\int_{t_0}^\infty \left[\int_s^\infty \left[\tfrac{1}{\vartheta^6}\int_\delta^\infty \eta v^2 dv\right] d\vartheta\right] ds \\ = &\infty. \end{aligned}$$

Also, if

$$\begin{aligned} &\int_{t_0}^\infty \left[\beta\varsigma(s)\left(\int_s^\infty \int_u^\infty x(\chi)\,d\chi du\right)^m - \tfrac{m^{m+1}\int_s^\infty x(\chi)d\chi}{(m+1)^{m+1}\int_s^\infty \int_u^\infty x(\chi)d\chi du}\right] ds \\ = &\int_{t_0}^\infty \left(\frac{\eta s^2}{60s^3} - \frac{60s^3}{4\,(20s^4)}\right) ds = \int_{t_0}^\infty \frac{1}{s}\left(\frac{\eta}{60} - \frac{3}{4}\right) ds \\ = &\infty,\ \text{for } \eta > 45 \end{aligned}$$

and

$$\begin{aligned} &\int_{t_0}^\infty \left[\beta\varsigma(s)\left(\frac{\varepsilon_2}{2}\zeta^2(s)\right)^m x^m(s) - \frac{m^{m+1}}{(m+1)^{m+1}\ x(s)\,h^{\frac{1}{m}}(s)}\right] ds \\ = &\int_{t_0}^\infty \frac{1}{s}\left(\frac{\eta\varepsilon_2}{10} - \frac{5}{4}\right) ds \\ = &\infty,\ \text{for } \eta > 45,\ \varepsilon_2 = \frac{5}{18} \in (0,\ 1). \end{aligned}$$

Consequently, every solution to equation (29) are oscillatory for $\eta > 45$ according to Theorem (1).

4. CONCLUSIONS

This work establishes certain oscillation requirements for a class of nonlinear delay differential equations of fourth-order. The study primary contribution is the derivation of new, more broadly applicable adequate criteria that build upon previously published findings. An effective analytical framework for analyzing the oscillatory nature of solutions is provided by the used methodology, which is based on a generalized Riccati transformation and enables the conversion of the original fourth-order neutral differential equation into an equivalent system of first-order differential inequalities. These findings add to the theoretical knowledge of neutral-type equations and present adaptable methods that can be used with a greater variety of functional differential systems.
In order to provide more thorough oscillation criteria for future studies, it might be interesting to loosen some of the current assumptions. Additionally, applying the current findings to odd-order equations and investigating different neutral structures like

$$\left[b\left(t\right)\left(\left(x\left(t\right)+p(t)x\left(\vartheta\left(t\right)\right)\right)'''\right)^{\gamma}\right]'+q\left(t\right)f\left(x\left(\tau\left(t\right)\right)\right)=0$$

might offer more profound understanding of the dynamics of neutral term nonlinear systems.

Author Contribution

- Conceptualization; Data curation; Formal analysis: N. Al-Zawqari
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