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Research Article

Analysis of Ricci Inheritance Collineations for Cylindrically Symmetric Spacetimes with Degenerate Ricci Tensor

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ABSTRACT

This research explores the Ricci Inheritance Collineations (RICs) within cylindrically symmetric Marder type spacetimes. We categorize these spacetimes based on their RICs, examining both degenerate and non-degenerate Ricci tensor configurations. Through solving a system of coupled partial differential equations across thirteen distinct scenarios, we provide a thorough examination of inheritance symmetries. Our findings demonstrate the existence of infinite-dimensional RICs in multiple cases, offering valuable insights into the geometric and physical characteristics of these spacetimes.

1. INTRODUCTION

Within the field of general relativity, categorizing spacetimes based on their symmetry characteristics stands as a vital research direction. The most elementary among these symmetries are Killing symmetries, alternatively termed isometries. These particular symmetries are defined through vector fields that preserve the metric tensor under Lie transport. The mathematical formulation of these symmetries is captured by the Killing equation:

$$g_{ab,c}\xi^c + g_{bc}\xi^c_{,a} + g_{ac}\xi^c_{,b} = 0, \tag{1}$$

where g_{ab} represents the metric tensor components, ξ^c denotes the components of the Killing vector field, and commas in the subscript indicate partial derivatives with respect to spacetime coordinates. When the right-hand side of Eq. (1) is modified to $2\alpha g_{ab}$ (with α being a constant), we obtain what are known as homothetic symmetries.

In the context of Einstein field equations, the metric tensor characterizes the geometric structure of a spacetime, whereas the Ricci tensor reveals its physical attributes. By replacing the metric tensor

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g_{ab} with the Ricci tensor R_{ab} in our symmetry analysis, we arrive at the concept of Ricci collineations (RCs). According to the framework established by Katzin et al. [1], these collineations satisfy the following equation:

$$R_{ab,c}\xi^c + R_{bc}\xi^c_{,a} + R_{ac}\xi^c_{,b} = 0. \tag{2}$$

Modifying Eq. (2) by replacing the zero on the right-hand side with $2\alpha R_{ab}$ (where α remains constant) yields:

$$R_{ab,c}\xi^c + R_{bc}\xi^c_{,a} + R_{ac}\xi^c_{,b} = 2\alpha R_{ab}. \tag{3}$$

Under these conditions, the vector field ξ is designated as either a homothetic RC or Ricci inheritance symmetry [2, 3]. Should α in Eq. (3) instead represent an arbitrary function dependent on all spacetime coordinates, ξ then qualifies as a conformal RC.

RCs command considerable attention among spacetime symmetries, primarily because while every symmetry qualifies as an RC, the reverse relationship does not necessarily hold. Despite their apparent similarity, Eqs. (1) and (2) embody fundamentally distinct mathematical principles. The metric tensor g_{ab} in Eq. (1) must satisfy the conditions of being non-singular and finite, which translates to $det\ g_{ab} \neq 0$. In contrast, the Ricci tensor R_{ab} in Eq. (2) may exhibit singularities but must remain finite [4]. Further differentiating these concepts is the fact that the collection $K(M)$ of all Killing vectors for a spacetime manifold M constitutes a finite-dimensional Lie algebra, with $dim\ K(M) \leq 10$ being a well-established constraint. Conversely, the set $RC(M)$ encompassing all RCs for a spacetime manifold M forms a vector space whose dimensionality may be finite or infinite, and which may or may not qualify as a Lie algebra. In scenarios where the Ricci tensor demonstrates non-singularity (non-degeneracy), indicated by $det\ g_{ab} \neq 0$, the set $RC(M)$ organizes into a finite-dimensional Lie algebra. However, when the Ricci tensor becomes singular (degenerate), as characterized by $det\ g_{ab} = 0$, the set $RC(M)$ generates a vector space that could be either infinite or finite-dimensional, potentially failing to form a Lie algebra [4].

Although Ricci collineations (RCs) have been thoroughly explored across diverse spacetime geometries, Ricci inheritance symmetries remain relatively underdeveloped in the existing literature. This work specifically focuses on investigating these Ricci inheritance symmetries. Several researchers have contributed to this field: [6] classified Bianchi type V spacetimes containing perfect-fluid matter according to their RCs, while [5] categorized static spherically symmetric spacetimes based on RCs and established connections between isometries and RCs. The identification of RCs for Bianchi type I, III, and Kantowski-Sachs spacetimes was reported in [7], whereas RCs corresponding to Bianchi type II, VIII, and IX spacetimes were detailed in [8].

Further contributions include [9], which presented a comprehensive classification of cylindrically symmetric static Lorentzian manifolds based on RCs alongside comparisons with Killing and homothetic symmetries. Both RCs and RICs for non-degenerate FRW spacetimes were examined in [10], while the interconnections between RCs and isometries in static plane symmetric spacetimes were elucidated in [11]. Analyses of RCs for maximally symmetric transverse spacetimes, covering both degenerate and non-degenerate cases, were conducted in [12] and [13].

Discussions on Ricci inheritance symmetries in spherically symmetric spacetimes and Friedman models can be found in [14] and [15], respectively. The relationship between Lie symmetries of the Ricci tensor and energy-momentum tensor in static cylindrically symmetric spacetime was explored in [16], while a concise treatment of Ricci and Matter collineations for non-degenerate locally rotationally symmetric spacetimes was provided in [17].

The principal objective of this article is to classify a particular form of Bianchi type I spacetime, as presented in equation (4), according to Ricci inheritance symmetries. This classification employs the Ricci inheritance collineation equation (3) with α considered as a constant parameter. Subsequent research by the authors will generalize this work by replacing the constant α with an arbitrary function of spacetime coordinates, thereby enabling classification according to conformal Ricci collineation. The paper is structured as follows: Section 2 outlines the derivation of ten coupled RIC equations. Following Sections address the solution of these equations for non-degenerate and degenerate cases, respectively, to determine the Ricci inheritance collineations. The final section provides a brief summary of the research.

2. MATHEMATICAL PRELIMINARIES AND FORMULATION

Consider a cylindrically symmetric Marder type spacetime described in standard coordinates (x^0, x^1, x^2, x^3) with the following metric structure:

$$ds^2 = A^2(t)[-dt^2 + dx^2] + B^2(t)dy^2 + C^2(t)dz^2, \tag{4}$$

where A, B, and C represent non-vanishing functions that depend exclusively on the temporal coordinate t.
For this cylindrically symmetric Marder type spacetime, the non-vanishing components of the Ricci tensor can be expressed as:

$$R_{00} = -\frac{A''}{A} - \frac{B''}{B} - \frac{C''}{C} + \frac{A'^2}{A^2} + \frac{A'^2B'C'}{A^2BC} = a(t) \tag{5}$$

$$R_{11} = -\frac{A''}{A} + \frac{A'^2}{A^2} + \frac{A'^2B'C'}{A^2BC} = b(t) \tag{6}$$

$$R_{22} = -\frac{BB''}{A^2} - \frac{BB'C'}{A^2C} = c(t) \tag{7}$$

$$R_{33} = -\frac{CC''}{A^2} - \frac{CC'B'}{A^2B} = d(t) \tag{8}$$

Here, primes denote derivatives with respect to t. By setting α as a constant in Eq. (3) and utilizing Eqs. (4) and (5), we derive the following system of ten coupled partial differential equations, known as RIC equations:

$$2a\xi^0_{,0} + a'\xi^0 = 2a\alpha, \tag{9}$$

$$a\xi^0_{,1} + b\xi^1_{,0} = 0, \tag{10}$$

$$a\xi^0_{,2} + c\xi^2_{,0} = 0, \tag{11}$$

$$a\xi^0_{,3} + d\xi^3_{,0} = 0, \tag{12}$$

$$2b\xi^1_{,1} + b'\xi^0 = 2b\alpha, \tag{13}$$

$$b\xi^1_{,2} + c\xi^2_{,1} = 0, \tag{14}$$

$$b\xi^1_{,3} + d\xi^3_{,1} = 0, \tag{15}$$

$$2c\xi^2_{,2} + c'\xi^0 = 2c\alpha, \tag{16}$$

$$c\xi^2_{,3} + d\xi^3_{,2} = 0, \tag{17}$$

$$2d\xi^3_{,3} + d'\xi^0 = 2d\alpha, \tag{18}$$

We now proceed to solve this system of equations to obtain the RICs for the case of a degenerate Ricci tensor. When the Ricci tensor is degenerate, i.e., $detR_{ab} = 0$, we must consider different Ricci tensor components as zero. By setting these components to zero, we examine various cases.

3. CASE I

In this scenario, we consider $a(t) \neq 0$ and $b(t) = c(t) = d(t) = 0$. Solving the aforementioned ten equations yields the following infinite set of RICs:

$$\begin{aligned} \xi^0 &= \frac{1}{\sqrt{a}}(\alpha \int \sqrt{a}dt + k_1), \\ \xi^i &= \xi^i(t,x,y,z), \;\; where \;\; i = 1,2,3. \end{aligned} \tag{19}$$

4. CASE II

Here, we examine $b(t) \neq 0$ and $a(t) = c(t) = d(t) = 0$. Solving the system of equations leads to two possibilities:
$b' = 0$ or $b' \neq 0$.
If $b' = 0$, implying b is constant, solving the equations produces the following infinite set of RICs:

$$\begin{aligned} \xi^1 &= \alpha x + k_1, \\ \xi^i &= \xi^i(t,x,y,z), \;\; where \;\; i = 0,2,3. \end{aligned} \tag{20}$$

If $b' \neq 0$, solving the equations yields the following infinite set of RICs:

$$\begin{aligned} \xi^0 &= \frac{b}{b'}[\alpha - G^1_x(x)], \quad \xi^1 = G^1(x) \\ \xi^i &= \xi^i(t,x,y,z), \;\; where \;\; i = 2,3, \end{aligned} \tag{21}$$

where $G^1(x)$ is an arbitrary function of x only.

5. CASE III

In this case, we take $c(t) \neq 0$ and $a(t) = b(t) = d(t) = 0$. Solving the system of equations presents two possibilities:
$c' = 0$ or $c' \neq 0$.
If $c' = 0$, implying c is constant, solving the equations results in the following infinite set of RICs:

$$\begin{aligned} \xi^2 &= \alpha y + k_1, \\ \xi^i &= \xi^i(t,x,y,z), \;\; where \;\; i = 0,1,3. \end{aligned} \tag{22}$$

If $c' \neq 0$, solving the equations yields the following infinite set of RICs:

$$\begin{aligned} \xi^0 &= \frac{c}{c'}[\alpha - G^1_y(y)], \quad \xi^2 = G^1(y) \\ \xi^i &= \xi^i(t,x,y,z), \;\; where \;\; i = 1,3, \end{aligned} \tag{23}$$

where $G^1(y)$ is an arbitrary function of y only.

6. CASE IV

For this case, we consider $d(t) \neq 0$ and $a(t) = b(t) = c(t) = 0$. Solving the system of equations leads to two possibilities:
$d' = 0$ or $d' \neq 0$.
If $d' = 0$, implying d is constant, solving the equations produces the following infinite set of RICs:

$$\begin{aligned} \xi^3 &= \alpha z + k_1, \\ \xi^i &= \xi^i(t,x,y,z), \quad where \;\; i = 0,1,2. \end{aligned} \tag{24}$$

If $d' \neq 0$, solving the equations yields the following infinite set of RICs:

$$\begin{aligned} \xi^0 &= \frac{d}{d'}[\alpha - G^1_z(z)], \qquad \xi^3 = G^1(z) \\ \xi^i &= \xi^i(t,x,y,z), \quad where \;\; i = 1,2, \end{aligned} \tag{25}$$

where $G^1(z)$ is an arbitrary function of z only.

7. CASE V

In this scenario, we examine $a(t), b(t) \neq 0$ and $c(t) = d(t) = 0$. Solving the system of equations presents two possibilities:
$H^1_x(x) = 0$ or $H^1_x(x) \neq 0$.
If $H^1_x(x) = 0$, implying $H^1(x) = k_1$, solving yields the following infinite set of RICs:

$$\begin{aligned} \xi^0 &= \frac{1}{\sqrt{a}}(\alpha \int \sqrt{a} dt + k_1), \;\; \xi^1 = k_2 x + k_3 \\ \xi^i &= \xi^i(t,x,y,z), \quad where \;\; i = 2,3. \end{aligned} \tag{26}$$

This solution is subject to the condition $b = \xi(\alpha \int \sqrt{a} dt + k_1)^{\frac{2(\alpha - k_2)}{\alpha}}$.
If $H^1_x(x) \neq 0$, solving the equations yields the following infinite set of RICs:

$$\begin{aligned} \xi^0 &= \frac{1}{\sqrt{a}}(\alpha \int \sqrt{a} dt + k_1 \sinh \sqrt{q} x) + k_2 \cosh \sqrt{q} x + k_3 \\ \xi^1 &= (k_4 - k_1 \int \frac{\sqrt{a}}{b} dt) \cosh \sqrt{q} x + (k_5 - k_2 \int \frac{\sqrt{a}}{b} dt) \sinh \sqrt{q} x + k_6 x + k_7 \\ \xi^i &= \xi^i(t,x,y,z), \quad where \;\; i = 2,3, \end{aligned} \tag{27}$$

where $k_1, k_2, k_3, k_4, k_5, k_6$ and k_7 are integration constants.

8. CASE VI

Here, we consider $a(t), c(t) \neq 0$ and $b(t) = d(t) = 0$. Solving the system of equations leads to two possibilities:
$H^1_y(y) = 0$ or $H^1_y(y) \neq 0$.
If $H^1_y(y) = 0$, implying $H^1(y) = k_1$, solving yields the following infinite set of RICs:

$$\begin{aligned} \xi^0 &= \frac{1}{\sqrt{a}}(\alpha \int \sqrt{a} dt + k_1), \;\; \xi^2 = k_2 y + k_3 \\ \xi^i &= \xi^i(t,x,y,z), \quad where \;\; i = 1,3. \end{aligned} \tag{28}$$

This solution is subject to the condition $c = \xi(\alpha \int \sqrt{a}dt + k_1)^{\frac{2(\alpha - k_2)}{\alpha}}$.
If $H_y^1(y) \neq 0$, solving the equations yields the following infinite set of RICs:

$$\begin{aligned}
\xi^0 &= \frac{1}{\sqrt{a}}(\alpha \int \sqrt{a}dt + k_1 \sinh \sqrt{q}y) + k_2 \cosh \sqrt{q}y + k_3 \\
\xi^2 &= (k_4 - k_1 \int \frac{\sqrt{a}}{c}dt) \cosh \sqrt{q}y + (k_5 - k_2 \int \frac{\sqrt{a}}{c}dt) \sinh \sqrt{q}y + k_6 y + k_7 \\
\xi^i &= \xi^i(t, x, y, z), \quad where \;\; i = 1, 3,
\end{aligned} \tag{29}$$

where $k_1, k_2, k_3, k_4, k_5, k_6$ and k_7 are integration constants.

9. CASE VII

In this case, we examine $a(t), d(t) \neq 0$ and $b(t) = c(t) = 0$. Solving the system of equations presents two possibilities:
$H_z^1(z) = 0$ or $H_z^1(z) \neq 0$.
If $H_z^1(z) = 0$, implying $H^1(z) = k_1$, solving yields the following infinite set of RICs:

$$\begin{aligned}
\xi^0 &= \frac{1}{\sqrt{a}}(\alpha \int \sqrt{a}dt + k_1), \;\; \xi^3 \;=\; k_2 z + k_3 \\
\xi^i &= \xi^i(t, x, y, z), \quad where \;\; i = 1, 2.
\end{aligned} \tag{30}$$

This solution is subject to the condition $d = \xi(\alpha \int \sqrt{a}dt + k_1)^{\frac{2(\alpha - k_2)}{\alpha}}$.
If $H_z^1(z) \neq 0$, solving the equations yields the following infinite set of RICs:

$$\begin{aligned}
\xi^0 &= \frac{1}{\sqrt{a}}(\alpha \int \sqrt{a}dt + k_1 \sinh \sqrt{q}z) + k_2 \cosh \sqrt{q}z + k_3 \\
\xi^3 &= (k_4 - k_1 \int \frac{\sqrt{a}}{d}dt) \cosh \sqrt{q}z + (k_5 - k_2 \int \frac{\sqrt{a}}{d}dt) \sinh \sqrt{q}z + k_6 z + k_7 \\
\xi^i &= \xi^i(t, x, y, z), \quad where \;\; i = 1, 2,
\end{aligned} \tag{31}$$

where $k_1, k_2, k_3, k_4, k_5, k_6$ and k_7 are integration constants.

10. CASE VIII

Here, we consider $b(t), c(t) \neq 0$ and $a(t) = d(t) = 0$. Solving the system of equations sequentially reveals three possibilities:
$(i)\ (\frac{b}{c})' \neq 0,\ M_y^1(x, y) = 0,\ (ii)\ (\frac{b}{c})' = 0,\ M_y^1(x, y) \neq 0,\ (iii)\ (\frac{b}{c})' = 0,\ M_y^1(x, y) = 0.$

10.1 Case VIII (i)

If $(\frac{b}{c})' \neq 0$ and $M_y^1(x, y) = 0$, solving the remaining equations leads to three sub-cases:
$(a)\ (\frac{bc'}{b'c})' = 0,\ \alpha - k_1 \neq 0,\ (b)\ (\frac{bc'}{b'c})' \neq 0,\ \alpha - k_1 = 0,\ (c)\ (\frac{bc'}{b'c})' = 0,\ \alpha - k_1 = 0.$
Case VIII (i)(a): When $(\frac{bc'}{b'c})' = 0$, implying $\frac{bc'}{b'c} = k_2$, solving the equations yields the following infinite set of RICs:

$$\begin{aligned}
\xi^0 &= \frac{2b}{b'}(\alpha - k_1), \;\; \xi^1 \;=\; k_1 x + k_3, \\
\xi^2 &= k_4 y + k_5, \quad \xi^3 \;=\; \xi^3(t, x, y, z),
\end{aligned} \tag{32}$$

This solution is subject to the condition $\alpha = \frac{k_1 k_2 - k_4}{k_2 - k_1}$.
Case VIII (i)(b): When $\alpha - k_1 = 0$, solving the equations yields the following infinite set of RICs:

$$\begin{aligned} \xi^0 &= 0, & \xi^1 &= \alpha x + k_2, \\ \xi^2 &= \alpha y + k_3, & \xi^3 &= \xi^3(t,x,y,z), \end{aligned} \tag{33}$$

This solution is subject to the condition $\alpha = k_1$.
Case VIII (i)(c): When $(\frac{bc'}{b'c})' = 0$, implying $\frac{bc'}{b'c} = k_2$, and $\alpha - k_1 = 0$, solving the equations yields the following infinite set of RICs:

$$\begin{aligned} \xi^0 &= 0, & \xi^1 &= \alpha x + k_3 \\ \xi^2 &= \alpha y + k_4, & \xi^3 &= \xi^3(t,x,y,z), \end{aligned} \tag{34}$$

This solution is subject to the condition $\alpha = k_1$.

10.2 Case VIII (ii)

When $(\frac{b}{c})' = 0$, implying $\frac{b}{c} = k_2$, and $M^1_y(x,y) \neq 0$, solving the equations yields the following infinite set of RICs:

$$\begin{aligned} \xi^0 &= \frac{2c}{c'}\alpha, & \xi^1 &= k_1 xy + k_3 x + k_4 y + k_5, \\ \xi^2 &= -\frac{1}{2}k_1 x^2 - k_4 x + k_6, & \xi^3 &= \xi^3(t,x,y,z), \end{aligned} \tag{35}$$

10.3 Case VIII (iii)

When $(\frac{b}{c})' = 0$, implying $\frac{b}{c} = k_2$, and $M^1_y(x,y) = 0$, solving the equations yields the following infinite set of RICs:

$$\begin{aligned} \xi^0 &= \frac{2c}{c'}\alpha, & \xi^1 &= k_1 xy + k_3 x + k_4 y + k_5, \\ \xi^2 &= -\frac{1}{2}k_1 x^2 - k_4 x + k_6, & \xi^3 &= \xi^3(t,x,y,z), \end{aligned} \tag{36}$$

11. CASE IX

In this scenario, we examine $b(t), d(t) \neq 0$ and $a(t) = c(t) = 0$. Solving the system of equations sequentially reveals three possibilities:
(i) $(\frac{b}{d})' \neq 0,\ M^1_z(x,z) = 0$, (ii) $(\frac{b}{d})' = 0,\ M^1_z(x,z) \neq 0$, (iii) $(\frac{b}{d})' = 0,\ M^1_z(x,z) = 0$.

11.1 Case IX (i)

If $(\frac{b}{d})' \neq 0$ and $M^1_z(x,z) = 0$, solving the remaining equations leads to three sub-cases:
(a) $(\frac{bd'}{b'd})' = 0,\ \alpha - k_1 \neq 0$, (b) $(\frac{bd'}{b'd})' \neq 0,\ \alpha - k_1 = 0$, (c) $(\frac{bd'}{b'd})' = 0,\ \alpha - k_1 = 0$.
Case IX (i)(a): When $(\frac{bd'}{b'd})' = 0$, implying $\frac{bd'}{b'd} = k_2$, solving the equations yields the following infinite set of RICs:

$$\begin{aligned} \xi^0 &= \frac{2b}{b'}(\alpha - k_1), & \xi^1 &= k_1 x + k_3, \\ \xi^3 &= k_4 z + k_5, & \xi^2 &= \xi^2(t,x,y,z), \end{aligned} \tag{37}$$

This solution is subject to the condition $\alpha = \frac{k_1k_2-k_4}{k_2-k_1}$.
Case IX (i)(b): When $\alpha - k_1 = 0$, solving the equations yields the following infinite set of RICs:

$$\begin{aligned} \xi^0 &= 0, & \xi^1 &= \alpha x + k_2, \\ \xi^3 &= \alpha z + k_3, & \xi^2 &= \xi^2(t,x,y,z), \end{aligned} \tag{38}$$

This solution is subject to the condition $\alpha = k_1$.
Case IX (i)(c): When $(\frac{bd'}{b'd})' = 0$, implying $\frac{bd'}{b'd} = k_2$, and $\alpha - k_1 = 0$, solving the equations yields the following infinite set of RICs:

$$\begin{aligned} \xi^0 &= 0, & \xi^1 &= \alpha x + k_3 \\ \xi^3 &= \alpha z + k_4, & \xi^2 &= \xi^2(t,x,y,z), \end{aligned} \tag{39}$$

This solution is subject to the condition $\alpha = k_1$.

11.2 Case IX (ii)

When $(\frac{b}{d})' = 0$, implying $\frac{b}{d} = k_2$, and $M^1_z(x,z) \neq 0$, solving the equations yields the following infinite set of RICs:

$$\begin{aligned} \xi^0 &= \frac{2d}{d'}\alpha, & \xi^1 &= k_1xz + k_3x + k_4z + k_5, \\ \xi^3 &= -\frac{1}{2}k_1x^2 - k_4x + k_6, & \xi^2 &= \xi^2(t,x,y,z), \end{aligned} \tag{40}$$

11.3 Case IX (iii)

When $(\frac{b}{d})' = 0$, implying $\frac{b}{d} = k_2$, and $M^1_z(x,z) = 0$, solving the equations yields the following infinite set of RICs:

$$\begin{aligned} \xi^0 &= \frac{2d}{d'}\alpha, & \xi^1 &= k_1xz + k_3x + k_4z + k_5, \\ \xi^3 &= -\frac{1}{2}k_1x^2 - k_4x + k_6, & \xi^2 &= \xi^2(t,x,y,z), \end{aligned} \tag{41}$$

12. CASE X

Here, we consider $c(t), d(t) \neq 0$ and $a(t) = b(t) = 0$. Solving the system of equations sequentially reveals three possibilities:
(i) $(\frac{c}{d})' \neq 0,\ M^1_z(y,z) = 0$, (ii) $(\frac{c}{d})' = 0,\ M^1_z(y,z) \neq 0$, (iii) $(\frac{c}{d})' = 0,\ M^1_z(y,z) = 0$.

12.1 Case X (i)

If $(\frac{c}{d})' \neq 0$ and $M^1_z(y,z) = 0$, solving the remaining equations leads to three sub-cases:
(a) $(\frac{cd'}{c'd})' = 0,\ \alpha - k_1 \neq 0$, (b) $(\frac{cd'}{c'd})' \neq 0,\ \alpha - k_1 = 0$, (c) $(\frac{cd'}{c'd})' = 0,\ \alpha - k_1 = 0$.
Case X (i)(a): When $(\frac{cd'}{c'd})' = 0$, implying $\frac{cd'}{c'd} = k_2$, solving the equations yields the following infinite set of RICs:

$$\begin{aligned} \xi^0 &= \frac{2c}{c'}(\alpha - k_1), & \xi^2 &= k_1y + k_3, \\ \xi^3 &= k_4z + k_5, & \xi^1 &= \xi^1(t,x,y,z), \end{aligned} \tag{42}$$

This solution is subject to the condition $\alpha = \frac{k_1k_2-k_4}{k_2-k_1}$.
Case X (i)(b): When $\alpha - k_1 = 0$, solving the equations yields the following infinite set of RICs:

$$\begin{aligned} \xi^0 &= 0, & \xi^2 &= \alpha y + k_2, \\ \xi^3 &= \alpha z + k_3, & \xi^1 &= \xi^1(t,x,y,z), \end{aligned} \tag{43}$$

This solution is subject to the condition $\alpha = k_1$.
Case X (i)(c): When $(\frac{cd'}{c'd})' = 0$, implying $\frac{cd'}{c'd} = k_2$, and $\alpha - k_1 = 0$, solving the equations yields the following infinite set of RICs:

$$\begin{aligned} \xi^0 &= 0, & \xi^2 &= \alpha y + k_3 \\ \xi^3 &= \alpha z + k_4, & \xi^1 &= \xi^1(t,x,y,z), \end{aligned} \tag{44}$$

This solution is subject to the condition $\alpha = k_1$.

12.2 Case X (ii)

When $(\frac{c}{d})' = 0$, implying $\frac{c}{d} = k_2$, and $M^1_z(y,z) \neq 0$, solving the equations yields the following infinite set of RICs:

$$\begin{aligned} \xi^0 &= \frac{2c}{c'}\alpha, & \xi^2 &= k_1yz + k_3y + k_4z + k_5, \\ \xi^3 &= -\frac{1}{2}k_1y^2 - k_4y + k_6, & \xi^1 &= \xi^1(t,x,y,z), \end{aligned} \tag{45}$$

12.3 Case X (iii)

When $(\frac{c}{d})' = 0$, implying $\frac{c}{d} = k_2$, and $M^1_z(y,z) = 0$, solving the equations yields the following infinite set of RICs:

$$\begin{aligned} \xi^0 &= \frac{2c}{c'}\alpha, & \xi^2 &= k_1yz + k_3y + k_4z + k_5, \\ \xi^3 &= -\frac{1}{2}k_1y^2 - k_4y + k_6, & \xi^1 &= \xi^1(t,x,y,z), \end{aligned} \tag{46}$$

13. CASE XI

In this case, we examine $a(t), b(t), c(t) \neq 0$ and $d(t) = 0$. Solving the system of equations sequentially yields the following infinite set of RICs:

$$\begin{aligned} \xi^0 &= \frac{2c}{c'}\alpha, & \xi^1 &= \alpha x(1-k_1) + k_2, \\ \xi^2 &= 0, & \xi^3 &= \xi^3(t,x,y,z), \end{aligned} \tag{47}$$

This solution is subject to the conditions $\frac{b'c}{bc'} = k_1$ and $\frac{a'}{a} = \frac{c'}{c}(1 - 2(\frac{c}{c'})')$.

14. CASE XII

Here, we consider $a(t), b(t), d(t) \neq 0$ and $c(t) = 0$. Solving the system of equations sequentially yields the following infinite set of RICs:

$$\begin{aligned} \xi^0 &= \frac{2d}{d'}\alpha, & \xi^1 &= \alpha x(1-k_1) + k_2, \\ \xi^3 &= 0, & \xi^2 &= \xi^2(t,x,y,z), \end{aligned} \tag{48}$$

This solution is subject to the conditions $\frac{b'd}{bd'} = k_1$ and $\frac{a'}{a} = \frac{d'}{d}(1 - 2(\frac{d'}{d})')$.

15. CASE XIII

In this scenario, we examine $a(t), c(t), d(t) \neq 0$ and $b(t) = 0$. Solving the system of equations sequentially yields the following infinite set of RICs:

$$\begin{aligned} \xi^0 &= \frac{2d}{d'}\alpha, \quad \xi^2 = \alpha y(1 - k_1) + k_2, \\ \xi^3 &= 0, \qquad \xi^1 = \xi^1(t, x, y, z), \end{aligned} \tag{49}$$

This solution is subject to the conditions $\frac{c'd}{cd'} = k_1$ and $\frac{a'}{a} = \frac{d'}{d}(1 - 2(\frac{d'}{d})')$.

16. CONCLUSION

This study presents a comprehensive classification of cylindrically symmetric Marder type spacetimes with degenerate Ricci tensor based on their Ricci Inheritance Collineations (RICs). Through systematic analysis of thirteen distinct cases, we have derived the explicit forms of RIC vector fields and established the conditions under which these symmetries manifest. Our investigation reveals that degenerate Ricci tensor configurations in these spacetimes exhibit remarkably rich symmetry structures, with infinite-dimensional RICs emerging across multiple scenarios.

The principal contribution of this work lies in demonstrating that cylindrically symmetric spacetimes with degenerate Ricci tensors possess significantly more extensive symmetry algebras than previously documented in non-degenerate cases. The emergence of infinite-dimensional RICs in Cases I through XIII underscores the profound geometric complexity of these spacetimes and provides new insights into their intrinsic properties. Particularly noteworthy are the conditions identified on metric functions $A(t)$, $B(t)$, and $C(t)$ that give rise to specific symmetry structures, establishing concrete relationships between functional forms of metric components and resulting inheritance symmetries.

From a physical standpoint, these findings have important implications for understanding conservation laws and integrability properties in curved spacetimes. The infinite-dimensional nature of the RICs suggests the presence of hidden symmetries that may correspond to additional conservation laws beyond those associated with Killing vectors. This could potentially influence approaches to quantization and particle dynamics in these gravitational backgrounds. The degenerate Ricci tensor scenarios examined here are particularly relevant to physically motivated spacetimes including gravitational wave solutions and certain anisotropic cosmological models.

Several promising avenues for future investigation emerge from this research. The natural extension to conformal Ricci collineations, where the constant α in the inheritance equation becomes an arbitrary function of spacetime coordinates, would provide a more complete symmetry classification. Additionally, exploring connections between RICs and other geometric properties such as curvature invariants and energy conditions could yield deeper physical insights. Applying these symmetry techniques to specific astrophysical scenarios, including gravitational collapse models and cosmic string geometries, represents another valuable direction for further study.

In summary, this work significantly advances the understanding of inheritance symmetries in cylindrically symmetric spacetimes with degenerate Ricci tensors. The systematic classification presented here establishes a foundation for future research into the geometric and physical implications of these symmetry structures, while highlighting the rich mathematical structure inherent in these spacetime configurations.

AUTHOR'S CONTRIBUTION STATEMENT

I. Khan contributed to the conceptualization, conducted the investigation and wrote the original draft. A. Ali and S. Khan performed formal analysis, carried out validation. A. Azad developed the methodology and performed validation. B. Zada undertook formal analysis and participated in writing – review and editing. All authors discussed the results and contributed to the final manuscript.

DECLARATIONS

Conflict of Interest

The authors declare no conflict of interest.

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