Global Journal of Sciences

Global Journal of Sciences
Volume 2, Issue 2, 2025, Pages 86-93
DOI: https://doi.org/10.48165/gjs.2025.2207; ISSN: 3049-0456
https://acspublisher.com/journals/index.php/gjs

www.acspublisher.com

Research Article

Variational Iteration Method with an Auxiliary Parameter for Some Kinds of Partial Differential Equations

Waqar Amin[1], Tufail Ahmad Khan[1], Jutarat Kongson[2,*] Amir Muhammad[1]

[1]*Department of Basic Sciences, University of Engineering and Technology Peshawar, Pakistan*
[2]*Department of Mathematics, Faculty of Science, Burapha University, Chonburi 20131, Thailand*

ARTICLE INFO

Article History
Received 17 Jun 2025
Accepted 11 Sep 2025
Published 24 Sep 2025

Keywords
Modified variational iteration-method
Modified equal width equation
MVIA-I
Regulerized long wave equation

ABSTRACT

In this paper, we present the Modified Variational Iteration Algorithm-I (MVIA-I) for obtaining numerical solutions of the modified equal width (MEW) wave equation and the regularized long wave (RLW) equation. This recently developed technique introduces an auxiliary parameter that significantly accelerates the convergence of the series solutions. The method provides both approximate and exact solutions with easily computable terms for linear and nonlinear PDEs, without requiring Adomian polynomials, small perturbations, discretization, or linearization.
To evaluate its accuracy, reliability, and efficiency, the results obtained by the proposed algorithm are compared with those of the standard Variational Iteration Method (VIM) and the Reduced Differential Transform Method (RDTM). The comparison clearly shows that MVIA-I is computationally efficient, highly accurate, and superior to existing methods. Two numerical test problems are included to assess the performance of the modified algorithm, and the accuracy is measured by calculating the absolute errors for different parameter values. The results demonstrate that MVIA-I converges rapidly and provides solutions of excellent precision, confirming its effectiveness as a robust tool for solving nonlinear PDEs.

1. INTRODUCTION

Partial differential equations (PDEs) play a fundamental role in modeling a wide range of phenomena in science, engineering, and applied mathematics. They are extensively used to simulate physical processes in fields such as fluid dynamics, heat transfer, electromagnetics, and material science. Understanding the behavior of these processes requires solving complex mathematical problems. Although exact or analytical solutions are often desirable, they are usually feasible only for simple cases with well-defined boundary conditions. In most realistic scenarios, the challenges arise from both the mathematical complexity of nonlinear PDEs and the substantial computational resources required.

For nonlinear PDEs (NPDEs), which are frequently encountered in STEM disciplines, obtaining exact solutions is particularly difficult and often computationally inefficient. As a result, approximation schemes and numerical techniques have become indispensable tools for handling such equations. In recent years, the numerical treatment of PDEs has gained increasing importance, driven by advances in computational technology that have enabled significant progress in this area. The flexibility of numerical methods allows them to address a wide variety of nonlinear problems and provide solutions with acceptable accuracy, making them essential in both theoretical and applied contexts.

Among various approaches, the variational iteration method (VIM), introduced by Ji-Huan He [1] and later refined [2], has attracted considerable attention due to its simplicity and efficiency. Unlike many traditional methods, VIM does not require linearization or discretization of the given equations. By constructing correction functionals with optimally determined

* *Corresponding author. Email: jutarat_k@go.buu.ac.th*

Lagrange multipliers, VIM yields analytical series solutions that are both robust and effective. This makes VIM a widely adopted method for tackling nonlinear problems, and in many cases, it has been shown to outperform other established techniques.
In recent years, numerous techniques have been proposed for solving different classes of PDEs. For example, Raza et al. [3] applied the reduced Painlevé technique to study three coupled nonlinear Maccari models, while Kumar et al. [4] used the rational sine–Gordon expansion method to analyze a chiral nonlinear Schrödinger equation. Fadhal et al. [5] proposed new approaches for nonlinear fractional differential equations with beta derivatives, and Cohen [6] applied the homotopy decomposition method. Ahmad et al. [7–9] suggested modified variational iteration algorithms as well as a unified method [10] for such problems. Zafar et al. [11] employed three distinct approaches, whereas Khan et al. [12] used the local meshless collocation method for numerical evaluations of NPDEs. Similarly, Silem et al. [13] solved the (1+2)-dimensional nonlinear Schrödinger equation using a generalized algebraic method, and Islam and Akbulut [14] investigated the Biswas–Arshed equation with beta derivatives. Ismael et al. [15] examined the Schrödinger–Boussinesq system using beta derivatives.
Other notable contributions include Gurefe et al. [16], who applied the trial equation method to the KdV equation with dual-power-law nonlinearity; Nadeem et al. [17], who introduced the modified Laplace variational iteration method; Hosseini et al. [18], who used the exp-ϕ ε-expansion method; Kazmi et al. [19], who investigated bifurcation-phase portraits of the q-deformed Sinh–Gordon equation; Ozkan and Akar [20], who applied the sub-equation method; and Jawad et al. [21], who developed a modified simple equation method.
Despite the considerable progress, there remains a need for further research into the development and application of efficient analytical and numerical methods for solving NPDEs. In particular, extensions and improvements of VIM continue to provide promising avenues for handling challenging nonlinear problems across diverse fields, including physics, oceanography, meteorology, and aerospace engineering. The present study focuses on exploring such adaptations and applications of VIM, thereby contributing to ongoing efforts to advance the theory and practice of solving nonlinear PDEs.

2. MODIFIED VARIATIONAL ITERATION ALGORITHM-I (MVIA-I)

To illustrate the general concept of modified variational iteration algorithm-I, consider the following nonlinear differential equation.

$$L\left[u(x)\right] + N\left[u(x)\right] = c(x), \tag{1}$$

Here, $L\left[u(x)\right]$ represent the linear term, $N\left[u(x)\right]$ denotes the nonlinear term, and $c(x)$ is the source term. The approximate solution $u_{k+1}(x)$ of equation (1) with given initial condition $u_0(x)$, can be obtained by using the following correctional function:

$$u_{k+1}(x) = u_k(x) + \int_0^x \lambda\left(\psi\right)\left[L\left\{u_k\left(\psi\right)\right\} + N\left\{u_k\left(\psi\right)\right\} - c\left(\psi\right)\right] d\psi, \tag{2}$$

Here λ, is a parameter called the Lagrange multiplier [2], which can be determine by taking δ on both sides of the recurrence relation (2) w.r.t. the variable $u_k(x)$, resulting in:

$$\delta u_{k+1}(x) = \delta u_k(x) + h\delta \int_0^x \lambda\left(\psi\right)\left[L\left\{u_k\left(\psi\right)\right\} + N\left\{u_k\left(\psi\right)\right\} - c\left(\psi\right)\right] d\psi, \tag{3}$$

here $u_k\left(\psi\right)$, is restricted term, i.e., $\delta u_k\left(\psi\right) = 0$, and gives the following Lagrange multipliers:
$\lambda = -1$ for $b = 1$,
$\lambda = v - s$ for $b = 2$,
Additionally, the Lagrange multiplier in the cases when $b \geq 1$ is possible to obtained through the broad formula that as follows:

$$\lambda = \frac{(-1)^b (v-s)^{b-1}}{(b-1)!} \tag{4}$$

Following that, an iteration formula is created by utilizing the value of λ in the correctional function as (2).

$$u_{k+1}(x) = u_k(x) + \int_0^x \frac{(-1)^b (v-s)^{b-1}}{(b-1)!})\left[L\left\{u_k\left(\psi\right)\right\} + N\left\{u_k\left(\psi\right)\right\} - c\left(\psi\right)\right] d\psi, \tag{5}$$

Implementing the iterative formula (5) continuously after beginning using a suitable initial assumption yields the iterative approximations. The following is the exact solution $u(x)$ that is obtained:

$$u(x) = \lim_{k\to\infty} u_k(x) \tag{6}$$

In conclusion, the iterative formulation of the complete solution procedure for equation (1) is as follows:

$$\begin{cases} u_0(x) \text{ is a proper initial approximation} \\ u_1(x) = u_0(x) + \int\limits_0^x \lambda(\psi)[L\{u_0(\psi)\} + N\{u_0(\psi)\} - c(\psi)]d\psi \\ u_{k+1}(x) = u_k(x) + \int\limits_0^x \lambda(\psi)[L\{u_k(\psi)\} + N\{u_k(\psi)\} - c(\psi)]d\psi \\ n = 1, 2, ... \end{cases} \tag{7}$$

Basic Variational Iteration Algorithm-I is the procedure for obtaining approximation solutions that is described in (7). It is an enhanced iteration approach founded on the commonly used principle of Lagrange multipliers [2]. More and more issues in various applied sciences and engineering domains have seen the use of this strategy in recent years to provide analytical and numerical solutions [22]. This study primarily focuses on adding an auxiliary convergence term, "h," to the iteration formula VIA-I. This term keeps track of how quickly the estimated solution is convergent. Thus, algorithm (7) needs the refined form that follows:

$$\begin{cases} u_0(x) \text{ is a proper initial approximation} \\ u_1(x,h) = u_0(x) + h\int\limits_0^x \lambda(\psi)[L\{u_0(\psi)\} + N\{u_0(\psi)\} - c(\psi)]d\psi \\ u_{k+1}(x,h) = u_k(x,h) + h\int\limits_0^x \lambda(\psi)[L\{u_k(\psi,h)\} + N\{u_k(\psi,h)\} - c(\psi,h)]d\psi \\ n = 1, 2, ... \end{cases} \tag{8}$$

The MVIA-I method eliminates the need for both domain discretization and the linearization of differential equations. By treating the nonlinear terms as restricted variations, researchers can calculate the Lagrange multiplier to obtain either analytical or numerical solutions of the given equations. This method allows the results to be expressed in a sequential form. The variational iteration method described in Equation (8) reduces to the conventional VIA-I when $h = 1$.

In this framework, as $n \to \infty$, the approximate solutions converge to the exact ones. Hence, increasing the number of iterations leads to more accurate results. Equation (8) has two notable advantages: (i) it incorporates an auxiliary parameter h that can be optimally determined, and (ii) it requires only a small number of iterations to achieve high accuracy. Moreover, the iteration step n directly influences the value of h, and it can be observed that $h \to 1$ as $n \to \infty$. A practical criterion for selecting n and determining h is provided through illustrative examples.

It is also evident that implementing the modified variational iteration method is straightforward. To compute the Lagrange multiplier in nonlinear problems, the nonlinear terms are treated as restricted variations. Substituting the calculated multiplier into the correction functional directly yields a recurrence relation, which simplifies the solution process [1].

3. NUMERICAL EXAMPLES

In this section, we consider two test problems: the modified equal width (MEW) wave equation and the regularized long wave (RLW) equation, to evaluate the accuracy of the proposed method. The performance of the method is assessed for different values of the auxiliary parameter, and the results obtained are both encouraging and significant. Furthermore, the absolute errors are calculated and compared with those reported for other existing methods in the literature.

3.1 Modified Equal Width Equation

Consider the following modified equal width (MEW) equation

$$u_s + 3u^2 u_x - u_{xxs} = 0, \tag{9}$$

the initial condition

$$u(x,0) = sechx, \tag{10}$$

[15] provides the precise solution to Equation (9).

$$u(x,s) = sech(x - 0.5s). \tag{11}$$

We can create a correctional function for the problem (9) in order to determine the solution of the aforementioned Modified Equal Width (MEW) equation by (MVIA-I):

$$\begin{aligned} u_{n+1}(x,s,h) = u_n(x,s,h) + h\int_0^s \lambda(\psi)[\frac{\partial u_n(x,\psi,h)}{\partial(\psi)} + 3(u_n(x,\psi,h))^2\frac{\partial u_n(x,\psi,h)}{\partial(x)} \\ -\frac{\partial^3 u_n(x,\psi,h)}{\partial(x)^2\partial(s)}]d\psi. \end{aligned} \tag{12}$$

We find that $\lambda(\psi) = -1$ is the value of λ. The following outcome is obtained by substituting this value of $\lambda(\psi)$ into Equation (12):

$$\begin{aligned} u_{n+1}(x,s,h) = u_n(x,s,h) - h\int_0^s [\frac{\partial u_n(x,\psi,h)}{\partial(\psi)} + 3(u_n(x,\psi,h))^2\frac{\partial u_n(x,\psi,h)}{\partial(x)} \\ -\frac{\partial^3 u_n(x,\psi,h)}{\partial(x)^2\partial(s)}]d\psi. \end{aligned} \tag{13}$$

It begins with the procedure with the initial approximation $u(x,0) = sechx$, which is given by Equation (10). Using the preceding iteration formula (13), we can immediately obtain the subsequent approximations, which are:

$$u_0(x,0) = 1/cosh(x).$$

$$\begin{aligned} u_1(x,s,h) =& 1/cosh(x) - (24 * h * s * exp(3 * x))/(4 * exp(2 * x) + 6 * exp(4 * x) \\ &+ 4 * exp(6 * x) + exp(8 * x) + 1) + (24 * h * s * exp(5 * x))/ \\ &(4 * exp(2 * x) + 6 * exp(4 * x) + 4 * exp(6 * x) + exp(8 * x) + 1). \end{aligned} \tag{14}$$

$$\begin{aligned} u_2(x,s,h) =& 1/cosh(x) - (192 * h^2 * s * exp(3 * x))/(13 * exp(2 * x) \\ &+ 78 * exp(4 * x) + 286 * exp(6 * x) + 715 * exp(8 * x) \\ &+ 1287 * exp(10 * x) + 1716 * exp(12 * x) + 1716 * exp(14 * x) \\ &+ 1287 * exp(16 * x) + 715 * exp(18 * x) + 286 * exp(20 * x) \\ &+ 78 * exp(22 * x). \end{aligned} \tag{15}$$

After the second order approximation, we finish with procedure. To find the optimal values of the auxiliary parameter h, we define the Residual function for the closest solution [17].

$$r_2\,(x,\ s,\ h) = \frac{\partial u_2(x,\psi,h)}{\partial(\psi)} + 3(u_2(x,\psi,h))^2\frac{\partial u_2(x,\psi,h)}{\partial(x)} - \frac{\partial^2 u_2(x,\psi,h)}{\partial(x)^2\partial(s)}d\psi = 0. \tag{16}$$

The minimum values of the residual function mentioned previously occur at different values of h. The Matlab software is employed to determine the optimal value of the auxiliary parameter h. When this value is substituted into $u_2(x,s,h)$, the absolute errors of the second-order approximation of the proposed algorithm decrease significantly.

To validate the accuracy of the method, we compared our results with those reported in [23]. The findings confirm that the proposed algorithm outperforms both the variational iteration method [1] and the reduced differential transform method [23]. The Modified Variational Iteration Algorithm-I (MVIA-I) converges rapidly and yields highly accurate results for all considered parameter values, as demonstrated in Tables I and II. Figure 1 presents plots for different values of t and x, which illustrate the behavior of the modified equal width wave equation under various conditions.

TABLE I. Evaluation between the precise and numerical solutions for Test problem 1 with varying variables for x and a constant value.

s	x	h	Exact Solution	MVIM Result	VIM	RDTM [23]
0.00015	-0.5	0.54640	0.8867881464	0.88699506720	0.88764979199	0.8866738290
0.00015	0	0.51724	0.9999999971	0.99999997291	0.99999989871	0.9999998987
0.00015	0.5	0.54642	0.8868496181	0.88664269041	0.88598800718	0.8869639226
0.00015	0.10	0.52231	0.9950281840	0.99492462500	0.99458759468	0.9950648383

TABLE II. Absolute errors for Test trial 1 at various x values and constant s values.

s	x	Absolute error in MVIM	Absolute error in VIM	Absolute Error in RDTM
0.00015	-0.5	2.069207308441623e-04	8.616455239158771e-04	1.143173911899442e-04
0.00015	0	2.427574818941736e-08	9.843750004012719e-08	9.843750004012719e-08
0.00015	0.5	2.069281780188170e-04	8.616114203978675e-04	1.143040558891917e-04
0.00015	0.10	1.035590826123922e-04	4.405894060320881e-04	3.665428471522070e-05

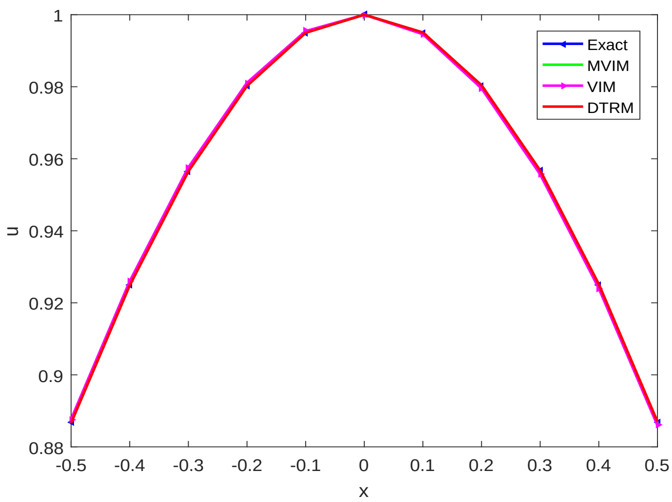

Fig. 1. Numerical solution for Test problem 1 at different values of x and fixed value of s

3.2 Regularized Long Wave Equation

Consider the following regularized long wave equation

$$u_s + u_x + uu_x - 0.125u_{xxs} = 0, \tag{17}$$

the initial condition

$$u(x, 0) = 3sech^2 x, \tag{18}$$

the exact solution of Equation(17) is given by [23],

$$u(x, s) = 3sech^2(x - 2s). \tag{19}$$

The above Regularized Long Wave equation can be solved using (MVIA-I) using the following steps: First, we can build the corrective function for equation (17).

$$\begin{aligned} u_{n+1}(x, s, h) = u_n(x, s, h) + h \int_0^s \lambda(\psi)[\frac{\partial u_n(x, \psi, h)}{\partial(\psi)} + \frac{\partial u_n(x, \psi, h)}{\partial(x)} + \\ u_n(x, \psi, h)\frac{\partial u_n(x, \psi, h)}{\partial(x)} - 0.15\frac{\partial^3 u_n(x, \psi, h)}{\partial(x)^2\partial(s)}]d\psi. \end{aligned} \tag{20}$$

The value of $\lambda(\psi)$ can be find through the theory of variational principles [19]. We determine the value of $\lambda(\psi)$, to be $\lambda(\psi) = -1$. Substituting this value of $\lambda(\psi)$ into Equation(20), yields the following result:

$$\begin{aligned} u_{n+1}(x,s,h) = u_n(x,s,h) - h \int_0^s [\frac{\partial u_n(x,\psi,h)}{\partial(\psi)} + \frac{\partial u_n(x,\psi,h)}{\partial(x)} + \\ u_n(x,\psi,h)\frac{\partial u_n(x,\psi,h)}{\partial(x)} - 0.15\frac{\partial^3 u_n(x,\psi,h)}{\partial(x)^2\partial(s)}]d\psi. \end{aligned} \tag{21}$$

It begins the procedure with an initial approximation $u(x,0) = 3sech^2 x$, which is Equation (3.2). Using the previously given iteration formula (21), we can immediately obtain the subsequent approximations, which are:

$$u_0(x,s) = 3/cosh(x)^2.$$

$$u_1(x,s,h) = 3/cosh(x)^2 + (6*h*s*sinh(x))/cosh(x)^3 + (18*h*s*sinh(x))/cosh(x)^5.$$

$$\begin{aligned} u_2(x,s,h) =& 3/cosh(x)^2 + (6*h^2*s^2)/cosh(x)^2 \\ &+ (63*h^2*s^2)/cosh(x)^4 + (72*h^2*s^2)/cosh(x)^6 \\ &- (189*h^2*s^2)/cosh(x)^8 + (24*h^3*s^3 \\ &*sinh(x))/cosh(x)^5 + (180*h^3*s^3*sinh(x))/cosh(x)^7 \\ &+ (144*h^3*s^3*sinh(x))/cosh(x)^9 \\ &- (540*h^3*s^3*sinh(x))/cosh(x). \end{aligned}$$

At the second-order approximation, we end the process. We define the Residual function for the approximate solution so that we can find the optimal values of the auxiliary parameter h.

$$\begin{aligned} r_2(x,\ s,\ h) = \frac{\partial u_2(x,\psi,h)}{\partial(\psi)} + \frac{\partial u_2(x,\psi,h)}{\partial(x)} + u_2(x,\psi,h)\frac{\partial u_2(x,\psi,h)}{\partial(x)} \\ -0.125\frac{\partial^3 u_2(x,\psi,h)}{\partial(x)^2\partial(s)}d\psi = 0. \end{aligned}$$

The minimum values of the residual function mentioned previously occur at different values of h. The Matlab software is used to compute the optimal value of the auxiliary parameter h. The absolute errors of the second-order approximation of the proposed algorithm decrease significantly when this value of h is substituted into $u_2(x,s,h)$.

To validate the accuracy of the proposed approach, we compared our results with those reported in [23]. The comparison clearly shows that our method outperforms both the reduced differential transform method [23] and the variational iteration method [1]. The Modified Variational Iteration Algorithm-I (MVIA-I) converges rapidly and yields highly accurate results for all considered parameter values, as demonstrated in Tables III and IV. Figures 2 further illustrate the behavior of the modified equal width wave equation at different values of s and x.

TABLE III. Considering the exact and numerical solutions for Test trial 2 with varying 'x' values and a constant s value.

t	x	h	Exact Solution	MVIM Result	VIM	RDTM [23]
0.001	-5	1.50032	5.42575242e-04	5.427069020e-04	5.43115200e-04	5.436602457e-04
0.001	-4	1.52800	0.00400680350	0.00400754985	0.00401071904	0.00401478353
0.001	-3	1.73638	0.02948053750	0.02947406008	0.02950552758	0.02953750052
0.001	-2	2.07329	0.21113667664	0.21069284483	0.21114665818	0.21145775755
0.001	-1	0.57147	1.25608855720	1.25613246588	1.25482064167	1.25559813233
0.001	0	0.60666	2.999988000032	2.99998233396	2.99995200000	2.99995200000

TABLE IV. Absolute errors for Test trial 2 at various x values and constant t values.

s	x	Absolute error in MVIM	Absolute error in VIM	Absolute Error in RDTM
0.001	-5	1.316591554274002e-07	5.399575764924135e-07	1.085002880683601e-06
0.001	-4	7.463434742935315e-07	3.915537848297318e-06	7.980026463633681e-06
0.001	-3	6.477422586179116e-06	2.499008203666123e-05	5.696301926433681e-05
0.001	-2	4.438318118354656e-04	9.981544266735032e-06	3.210809167610018e-04
0.001	-1	4.390868056747266e-05	0.001267915520838	4.904248646904197e-04
0.001	0	5.666068267018432e-06	3.600003200077140e-05	3.600003200032731e-05

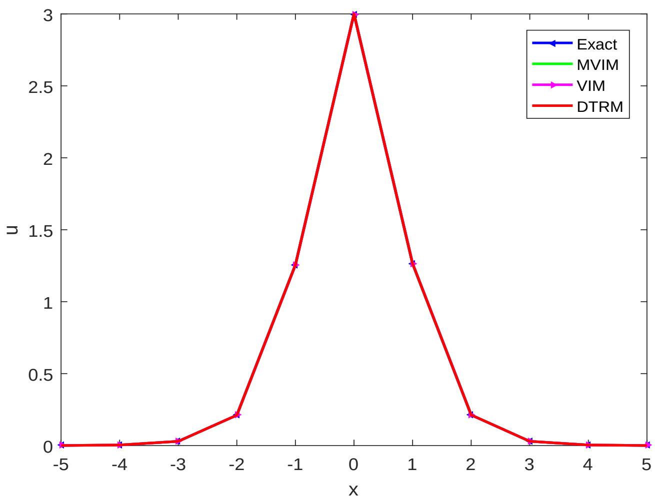

Fig. 2. Numerical solution for Test problem 1 at different values of x and fixed value of s

In this paper, we used the modified variational iteration algorithm-I for the numerical solution of modified equal width wave equation and regularized long wave equation. The technique effectively gives extremely precise solutions using various values of auxiliary parameter. This modified algorithm makes easy the computational work for solving linear and nonlinear problems arises in science and engineering, and results of high degree accuracy can be obtained in a few iterations as compared to earlier methods. The modified algorithm can be utilized without any need for discretization, linearization and lengthy calculations. We conclude that the proposed algorithm provides an accurate numerical/analytical solutions and can handle the nonlinear PDEs in a good and reliable manner in various cases. Author Contributions: Haider Ali: Writing Original Draft, Review; Muhammad Amjad: Conceptualization; Tehreem: Software; Shah Jahan: Conceptualization; Muhammad Zaman: Validation; Aamir Hussain Khan: Validation; Muhammad Ramzan: Writing Original Draft.

ACKNOWLEDGMENT

Not Applicable

DECLARATION

Declaration of Interest
The authors declare no conflict of interest

Data Availability Statement
This manuscript has no associated data.

Funding
There is no funding associated with this manuscript.

REFERENCES

[1] He, J.-H. (1999). Variational iteration method–a kind of non-linear analytical technique: some examples. *International Journal of Non-Linear Mechanics*, 34(4), 699–708.
[2] Rockafellar, R. T. (1993). Lagrange multipliers and optimality. *SIAM Review*, 35(2), 183–238.
[3] Raza, N., Arshed, S., & Asghar, M. H. (2025). Traveling wave solutions with modulation instability of coupled nonlinear Schrödinger equations via two analytical approaches. *International Journal of Theoretical Physics*, 64(9), 1–26.
[4] Kumar, D., Hosseini, K., & Samadani, F. (2017). The sine-gordon expansion method to look for the traveling wave solutions of the tzitzéica type equations in nonlinear optics. *Optik*, 149, 439–446.
[5] Fadhal, E., Ganie, A. H., Alharthi, N. S., Khan, A., Fathima, D., & Elamin, A. E. (2024). On the analysis and deeper properties of the fractional complex physical models pertaining to nonsingular kernels. *Scientific Reports*, 14(1), 22182.
[6] Cohen, J. M. (1968). The decomposition of stable homotopy. *Annals of Mathematics*, 87(2), 305–320.
[7] Ahmad, H., Khan, T. A., Stanimirovic, P. S., Shatanawi, W., & Botmart, T. (2022). New approach on conventional solutions to nonlinear partial differential equations describing physical phenomena. *Results in Physics*, 41, 105936.
[8] Ahmad, H., & Khan, T. A. (2020). Variational iteration algorithm I with an auxiliary parameter for the solution of differential equations of motion for simple and damped mass–spring systems. *Noise & Vibration Worldwide*, 51(1–2), 12–20.
[9] Ahmad, H., Seadawy, A. R., Khan, T. A., & Thounthong, P. (2020). Analytic approximate solutions for some nonlinear parabolic dynamical wave equations. *Journal of Taibah University for Science*, 14(1), 346–358.
[10] Ahmad, H., Alam, M. N., Rahim, M. A., Alotaibi, M. F., & Omri, M. (2021). The unified technique for the nonlinear time-fractional model with the beta-derivative. *Results in Physics*, 29, 104785.
[11] Zafar, Z. A., Rehan, K., Mushtaq, M., & Rafiq, M. (2017). Numerical modeling for nonlinear biochemical reaction networks. *Iranian Journal of Mathematical Chemistry*, 8(4), 413–423.
[12] Khan, M. N., Ahmad, I., Akgül, A., Ahmad, H., & Thounthong, P. (2021). Numerical solution of time-fractional coupled Korteweg–de Vries and Klein–Gordon equations by local meshless method. *Pramana*, 95(1), 6.
[13] Silem, A., Zhang, C., & Zhang, D. J. (2019). Dynamics of three nonisospectral nonlinear Schrödinger equations. *Chinese Physics B*, 28(2), 020202.
[14] Akbulut, A., & Islam, S. R. (2022). Study on the Biswas–Arshed equation with the beta time derivative. *International Journal of Applied and Computational Mathematics*, 8(4), 167.
[15] Ismael, H. F., Bulut, H., Baskonus, H. M., & Gao, W. (2021). Dynamical behaviors of the coupled Schrödinger-Boussinesq system with the beta derivative. *AIMS Mathematics*, 6(7), 7909–7928.
[16] Gurefe, Y., Sonmezoglu, A., & Misirli, E. (2011). Application of the trial equation method for solving some nonlinear evolution equations arising in mathematical physics. *Pramana*, 77(6), 1023–1029.
[17] Nadeem, M., Li, F., & Ahmad, H. (2019). Modified Laplace variational iteration method for solving fourth-order parabolic partial differential equations with variable coefficients. *Computers & Mathematics with Applications*, 78(6), 2052–2062.
[18] Hosseini, K., Bekir, A., & Ansari, R. (2017). Exact solutions of nonlinear conformable time-fractional Boussinesq equations using the exp-ϕ ε-expansion method. *Optical and Quantum Electronics*, 49(4), 131.
[19] Kazmi, S. S., Jhangeer, A., Raza, N., Alrebdi, H. I., Abdel-Aty, A. H., & Eleuch, H. (2023). The analysis of bifurcation, quasi-periodic, and soliton patterns for the new form of the generalized q-deformed Sinh-Gordon equation. *Symmetry*, 15(7), 1324.
[20] Özkan, E. M., & Akar, M. (2022). Analytical solutions of (2+1)-dimensional time conformable Schrödinger equation using improved sub-equation method. *Optik*, 267, 169660.
[21] Jawad, A. J. M., Petkovic, M. D., & Biswas, A. (2010). Modified simple equation method for nonlinear evolution equations. *Applied Mathematics and Computation*, 217(2), 869–877.
[22] Ahmad, H., Khan, T. A., Ahmad, I., Stanimirović, P. S., & Chu, Y. M. (2020). A new analyzing technique for nonlinear time-fractional Cauchy reaction-diffusion model equations. *Results in Physics*, 19, 103462.
[23] Keskin, Y., & Oturanç, G. (2010). Reduced differential transform method for solving linear and nonlinear wave equations.