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Research Article

Generalization of Hermite-Hadamard Integral Inequality via Caputo-Fabrizio Fractional Integral Operator

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ABSTRACT

Certainly, fractional calculus has grabbed the keen interest of numerous investigators in the last couple of decades. Numerous real-world problems and demonstrations have involved the development of the Caputo-Fabrizio fractional operator. Our article's primary goal is to enhance Hermite-Hadamard (H-H) integral inequality and its generalizations by applying the Caputo-Fabrizio fractional integral operator (CFFIO) and the preinvexity concept. We additionally offer some applications of our key findings to specific methods.

1. INTRODUCTION

Over the past century, numerous scientists have studied the mathematical notion of "convexity." In the advancement of various pure and applied science fields, this term has taken on a significant role and received remarkable attention from many scientists. Functional analysis, mathematical statistics, and financial mathematics all heavily rely on the theory of convexity. There are numerous real-world uses for convex function optimization, including modeling, controller design, and circuit design. Because of its many applications and significance, the term "convexity" has become a rich source of inspiration and a fascinating area of study for mathematicians and scientists. For a discussion of convexity and its properties, we encourage interested readers to consult the references [1–6].
The convexity property and the term "inequalities" are crucial to modern mathematical research. There is a close relationship between the two terms. The term "inequalities" has many applications in statistical problems, probability, functional analysis, mechanics, and numerical quadrature formulas. In this way, the hypothesis of inequalities could be considered a separate area of mathematical study. Readers who are interested ought to [7–11].
These days, fractional analysis and inequality theory have developed simultaneously. Over the past few decades, fractional calculus has grown in popularity and promise as a research area within the broad field of applied sciences. Newly introduced fractional derivatives and integrals with different viewpoints have been used by some mathematicians to analyze and resolve real-world issues in a variety of applied science domains. Joseph Liouville most likely provided

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the first logical definition of a fractional derivative. Between 1832 and 1837, he wrote about nine papers on fractional calculus, the last of which was published in 1855. N. H. Abel most likely used fractional calculus for the first time between 1802 and 1829.

2. PRELIMINARIES

We review some fundamental ideas needed for the article in this section.

Definition 2.1 ([12]). *Suppose $\mathcal{X} \subset \mathbb{R}^n$ be invex w.r.t. $\Upsilon(.,.)$, if*

$$\mathsf{e}_1 + \delta\Upsilon(\mathsf{e}_2, \mathsf{e}_1) \in \mathcal{A},$$

$\forall\ \mathsf{e}_1, \mathsf{e}_2 \in \mathcal{X}$ *and* $\delta \in [0, 1]$.

Definition 2.2 ([13]). *Assume that $\Upsilon : \mathcal{X} \times \mathcal{X} \times (0, 1] \to \mathbb{R}^n$ and $\mathcal{X} \subseteq \mathbb{R}^n$. Then $\mathcal{A}$ is m–invex w.r.t. Υ, if*

$$m\mathsf{e}_2 + \delta\Upsilon(\mathsf{e}_1, \mathsf{e}_2, m) \in \mathcal{A}$$

holds $\forall\ \mathsf{e}_1, \mathsf{e}_2 \in \mathcal{A}$, $m \in (0, 1]$ and $\delta \in [0, 1]$.

Example 2.1 ([13]). *Assume that $m = \frac{1}{4}$, $\mathcal{A} = [\frac{-\Upsilon}{2}, 0) \bigcup (0, \frac{1}{2}]$ and*

$$\Upsilon(\mathsf{e}_2, \mathsf{e}_1, m) = \begin{cases} m\cos(\mathsf{e}_2 - \mathsf{e}_1) \quad if\ \mathsf{e}_1 \in (0, \frac{\Upsilon}{2}], \mathsf{e}_2 \in (0, \frac{\Upsilon}{2}]; \\ -m\cos(\mathsf{e}_2 - \mu_1) \quad if\ \mathsf{e}_1 \in [\frac{-\Upsilon}{2}, 0), \mathsf{e}_2 \in [\frac{-\Upsilon}{2}, 0); \\ m\cos(\mathsf{e}_1) \quad if\ \mathsf{e}_1 \in (0, \frac{\Upsilon}{2}], \mathsf{e}_2 \in [\frac{-\Upsilon}{2}, 0); \\ -m\cos(\mathsf{e}_1) \quad if\ \mathsf{e}_1 \in [-\frac{\Upsilon}{2}, 0), \mathsf{e}_2 \in (0, \frac{\Upsilon}{2}]. \end{cases}$$

Then, $\mathcal{X}$ is m–invex set but not convex $\forall\ \eta \in [0, 1]$.

Definition 2.3 ([14]). *Suppose $\Upsilon : \mathcal{X} \times \mathcal{X} \to \mathbb{R}^n$ and $\mathcal{X} \subseteq \mathbb{R}^n$. Then an inequality in the following form*

$$\nabla(\mathsf{e}_2 + \eta\Upsilon(\mathsf{e}_1, \mathsf{e}_2)) \leq \eta\,\nabla(\mathsf{e}_1) + (1 - \eta)\,\nabla(\mathsf{e}_2), \qquad \forall \mathsf{e}_1,\ \mathsf{e}_2 \in \mathcal{A},\ \ \eta \in [0, 1],$$

is said to be preinvex.

Definition 2.4. *[15] Suppose $\Upsilon : \mathcal{X} \times \mathcal{X} \times (0, 1] \to \mathbb{R}^n$ and $\mathcal{X} \subseteq \mathbb{R}^n$. Then for every $\mathsf{e}_1, \mathsf{e}_2 \in \mathcal{X}$, $m \in (0, 1]$ and $\eta \in [0, 1]$, inequality*

$$\nabla(m\mathsf{e}_2 + \eta\Upsilon(\mathsf{e}_1, \mathsf{e}_2, m)) \leq \eta\nabla(\mathsf{e}_1) + m(1 - \eta)\nabla(\mathsf{e}_2), \tag{1}$$

is said to be generalized m-preinvex.

Condition *C*: Assume that $\mathcal{X} \subset \mathbb{R}$ is an open invex subset w.r.t. $\Upsilon : \mathcal{X} \times \mathcal{X} \to \mathbb{R}$, then $\mathsf{e}_1, \mathsf{e}_2 \in \mathcal{X}$ and $\eta \in [0, 1]$,

$$\begin{aligned} \Upsilon(\mathsf{e}_1, \mathsf{e}_1 + \eta\,\Upsilon(\mathsf{e}_2, \mathsf{e}_1)) &= -\eta\,\Upsilon(\mathsf{e}_2, \mathsf{e}_1) \\ \Upsilon(\mathsf{e}_2, \mathsf{e}_1 + \eta\,\Upsilon(\mathsf{e}_2, \mathsf{e}_1)) &= (1 - \eta)\ \Upsilon(\mathsf{e}_2, \mathsf{e}_1). \end{aligned}$$

For any $\mathsf{e}_1, \mathsf{e}_2 \in \mathcal{X}$, $\eta_1, \eta_2 \in [0, 1]$, then according to the above equations, we have

$$\Upsilon(\mathsf{e}_1 + \eta_2\,\Upsilon(\mathsf{e}_2, \mathsf{e}_1),\ \mathsf{e}_1 + \eta_1\,\Upsilon(\mathsf{e}_2, \mathsf{e}_1)) = (\eta_2 - \eta_1)\,\Upsilon(\mathsf{e}_2, \mathsf{e}_1).$$

This Condition is very crucial in the growth and optimization of the notion of inequalities (see [16, 17]).
Extended Condition C in the mood of m-preinvex function was also explored by Du et.al in [10].
Extended Condition *C*: Let $\mathcal{X} \subset \mathbb{R}$ an open invex subset w.r.t. $\Upsilon : \mathcal{X} \times \mathcal{X} \times (0, 1] \to \mathbb{R}$. Then Υ satisfied the Extended Condition C, if for any $\mathsf{e}_1, \mathsf{e}_2 \in \mathcal{X}$, $\eta \in [0, 1]$, we have

$$\begin{aligned} \Upsilon(\mathsf{e}_2, m\mathsf{e}_2 + \eta\,\Upsilon(\mathsf{e}_1, \mathsf{e}_2, m), m) &= -\eta\,\Upsilon(\mathsf{e}_1, \mathsf{e}_2, m) \\ \Upsilon(\mathsf{e}_1, m\mathsf{e}_2 + \eta\,\Upsilon(\mathsf{e}_1, \mathsf{e}_2, m), m) &= (1 - \eta)\ \Upsilon(\mathsf{e}_1, \mathsf{e}_2, m) \\ \Upsilon(\mathsf{e}_1, \mathsf{e}_2, m) &= -\Upsilon(\mathsf{e}_2, \mathsf{e}_1, m). \end{aligned}$$

Theorem 2.1. *[18] Suppose $p > 1$ and $\frac{1}{p} + \frac{1}{q} = 1$. Let $\Psi_1, \Psi_2 : [x_1, x_2] \to \mathbb{R}$ and $|\Psi_1|^p$ and $|\Psi_2|^q$ are integrable on $[x_1, x_2]$. Then*

$$\int_0^1 |\nabla_1(x)\nabla_2(x)|dx \leq \Big(\int_0^1 |\nabla_1(x)|^p dx\Big)^{\frac{1}{p}}\Big(\int_0^1 |\nabla_2(x)|^q dx\Big)^{\frac{1}{q}}.$$

Definition 2.5 ([19]). *Let $\mathcal{Q} \in H^1(\pi_1, \pi_2) = \{y \in L^2(\pi_1, \pi_2) : y' \in L^2(\pi_1, \pi_2)\}$, $\pi_1 < \pi_2$, $\alpha \in [0, 1]$, then the definition of the left fractional integral in the sense of C-F becomes*

$$^{CF}I^{\alpha}_{\pi_1}\mathcal{Q}(t) = \frac{1-\alpha}{B(\alpha)}\mathcal{Q}(t) + \frac{\alpha}{B(\alpha)}\int_{\pi_1}^{t}\mathcal{Q}(x)dx,$$

the right fractional integral

$$^{CF}I^{\alpha}_{\pi_2}\mathcal{Q}(t) = \frac{1-\alpha}{B(\alpha)}\mathcal{Q}(t) + \frac{\alpha}{B(\alpha)}\int_{t}^{b}\mathcal{Q}(x)dx,$$

where $B(\alpha) > 0$ is a normalization function satisfying $B(0) = B(1) = 1$.

3. H-H INEQUALITY VIA C-F FRACTIONAL OPERATOR

Theorem 3.1. *Assume that $\nabla : [m\mathsf{e}_1, m\mathsf{e}_1 + \Upsilon(\mathsf{e}_2, \mathsf{e}_1, m)] \to (0, \infty)$ is m-preinvex function on X^0 and $\nabla \in L[m\mathsf{e}_1, m\mathsf{e}_1 + \Upsilon(\mathsf{e}_2, \mathsf{e}_1, m)]$. If $\alpha \in [0, 1]$, then*

$$\begin{aligned}\nabla\left(\frac{2m\mathsf{e}_1 + \Upsilon(\mathsf{e}_2, \mathsf{e}_1, m)}{2}\right) &\leq \frac{B(\alpha)}{\alpha\Upsilon(\mathsf{e}_2, \mathsf{e}_1, m)}\left[\left(^{CF}_{m\mathsf{e}_1}I^{\alpha}\nabla\right)(k) + \left(^{CF}I^{\alpha}_{m\mathsf{e}_1+\Upsilon(\mathsf{e}_2,\mathsf{e}_1,m)}\nabla\right)(k) - \frac{2(1-\alpha)}{B(\alpha)}\nabla(k)\right] \\ &\leq \frac{\nabla(m\mathsf{e}_1) + \nabla(\mathsf{e}_2)}{2}, \end{aligned} \tag{2}$$

where $k \in [m\mathsf{e}_1, m\mathsf{e}_1 + \Upsilon(\mathsf{e}_2, \mathsf{e}_1, m)]$.

Proof. Since ∇ is a m-preinvex function on $[m\mathsf{e}_1, m\mathsf{e}_1 + \Upsilon(\mathsf{e}_2, \mathsf{e}_1, m)]$, we can write

$$\begin{aligned}2\nabla\left(\frac{2m\mathsf{e}_1 + \Upsilon(\mathsf{e}_2, \mathsf{e}_1, m)}{2}\right) &\leq \frac{2}{\Upsilon(\mathsf{e}_2, \mathsf{e}_1, m)}\int_{m\mathsf{e}_1}^{m\mathsf{e}_1+\Upsilon(\mathsf{e}_2,\mathsf{e}_1,m)}\nabla(x)dx \\ &= \frac{2}{\Upsilon(\mathsf{e}_2, \mathsf{e}_1, m)}\left(\int_{m\mathsf{e}_1}^{k}\nabla(x)dx + \int_{k}^{m\mathsf{e}_1+\Upsilon(\mathsf{e}_2,\mathsf{e}_1,m)}\nabla(x)dx\right).\end{aligned} \tag{3}$$

Multiplying inequality (3) with $\frac{\alpha\Upsilon(\mathsf{e}_2,\mathsf{e}_1,m)}{2B(\alpha)}$ and adding $\frac{2(1-\alpha)}{B(\alpha)}\nabla(k)$ both sides, we have

$$\begin{aligned}&\frac{2(1-\alpha)}{B(\alpha)}\nabla(k) + \frac{\alpha\Upsilon(\mathsf{e}_2, \mathsf{e}_1, m)}{B(\alpha)}\nabla\left(\frac{2m\mathsf{e}_1 + \Upsilon(\mathsf{e}_2, \mathsf{e}_1, m)}{2}\right) \\ &\leq \frac{2(1-\alpha)}{B(\alpha)}\nabla(k) + \frac{\alpha}{B(\alpha)}\left(\int_{m\mathsf{e}_1}^{k}\nabla(x)dx + \int_{k}^{m\mathsf{e}_1+\Upsilon(\mathsf{e}_2,\mathsf{e}_1,m)}\nabla(x)dx\right) \\ &= \left(\frac{(1-\alpha)}{B(\alpha)}\nabla(k) + \frac{\alpha}{B(\alpha)}\int_{m\mathsf{e}_1}^{k}\nabla(x)dx\right) + \left(\frac{(1-\alpha)}{B(\alpha)}\nabla(k) + \frac{\alpha}{B(\alpha)}\int_{k}^{m\mathsf{e}_1+\Upsilon(\mathsf{e}_2,\mathsf{e}_1,m)}\nabla(x)dx\right) \\ &= \left(^{CF}_{m\mathsf{e}_1}I^{\alpha}\nabla\right)(k) + \left(^{CF}I^{\alpha}_{m\mathsf{e}_1+\Upsilon(\mathsf{e}_2,\mathsf{e}_1,m)}\nabla\right)(k).\end{aligned} \tag{4}$$

This is the required proof of the 1^{st} inequality (2). For 2^{nd} inequality proof, we use

$$\frac{2}{\Upsilon(\mathsf{e}_2, \mathsf{e}_1, m)}\int_{m\mathsf{e}_1}^{m\mathsf{e}_1+\Upsilon(\mathsf{e}_2,\mathsf{e}_1,m)}\nabla(x)dx \leq \nabla(m\mathsf{e}_1) + \nabla(\mathsf{e}_2). \tag{5}$$

Using (3) in (5) to perform the same operation, we have

$$\left(^{CF}_{m\mathsf{e}_1}I^{\alpha}\nabla\right)(k) + \left(^{CF}I^{\alpha}_{m\mathsf{e}_1+\Upsilon(\mathsf{e}_2,\mathsf{e}_1,m)}\nabla\right)(k) \leq \frac{2(1-\alpha)}{B(\alpha)}\nabla(k) + \frac{\alpha\Upsilon(\mathsf{e}_2, \mathsf{e}_1, m)}{2B(\alpha)}\left(\nabla(m\mathsf{e}_1) + \nabla(\mathsf{e}_2)\right). \tag{6}$$

By recognising (6), the proof is completed.

4. GENERALIZATIONS OF H-H VIA C-F FRACTIONAL OPERATOR

Lemma 4.1. *Assume that* $\nabla : I = [m\mathrm{e}_1, m\mathrm{e}_1 + \Upsilon(\mathrm{e}_2, \mathrm{e}_1, m)] \to (0, \infty)$ *is differentiable function on* $X^0, \mathrm{e}_1, \mathrm{e}_2 \in X^0$ *with* $m\mathrm{e}_1 < m\mathrm{e}_1 + \Upsilon(\mathrm{e}_2, \mathrm{e}_1, m)$ *if* $\nabla' \in L[m\mathrm{e}_1, m\mathrm{e}_1 + \Upsilon(\mathrm{e}_2, \mathrm{e}_1, m)]$*, then*

$$\begin{aligned}&\frac{\Upsilon(\mathrm{e}_2, \mathrm{e}_1, m)}{2}\int_0^1 (1-2\eta)\nabla'(m\mathrm{e}_1 + \eta\Upsilon(\mathrm{e}_2, \mathrm{e}_1, m))d\eta + \frac{2(1-\alpha)}{\alpha\Upsilon(\mathrm{e}_2, \mathrm{e}_1, m)}\nabla(k)\\ &= -\frac{\nabla(m\mathrm{e}_1) + \nabla(m\mathrm{e}_1 + \Upsilon(\mathrm{e}_2, \mathrm{e}_1, m))}{2} + \frac{B(\alpha)}{\alpha\Upsilon(\mathrm{e}_2, \mathrm{e}_1, m)}\left[\left({}^{CF}_{m\mathrm{e}_1}I^{\alpha}\nabla\right)(k) + \left({}^{CF}I^{\alpha}_{m\mathrm{e}_1+\Upsilon(\mathrm{e}_2,\mathrm{e}_1,m)}\nabla\right)(k)\right],\end{aligned}$$

where $k \in [m\mathrm{e}_1, m\mathrm{e}_1 + \Upsilon(\mathrm{e}_2, \mathrm{e}_1, m)]$.

Proof. It is easy to see that

$$\begin{aligned}&\int_0^1 (1-2\eta)\nabla'(m\mathrm{e}_1 + \eta\Upsilon(\mathrm{e}_2, \mathrm{e}_1, m))d\eta\\ &= -\frac{\nabla(m\mathrm{e}_1) + \nabla(m\mathrm{e}_1 + \Upsilon(\mathrm{e}_2, \mathrm{e}_1, m))}{2} + \frac{2}{(\Upsilon(\mathrm{e}_2, \mathrm{e}_1, m))^2}\left(\int_{m\mathrm{e}_1}^{k}\nabla(x)dx + \int_{k}^{m\mathrm{e}_1+\Upsilon(\mathrm{e}_2,\mathrm{e}_1,m)}\nabla(x)dx\right).\end{aligned}$$

By multiplying both sides with $\frac{\alpha(\Upsilon(\mathrm{e}_2,\mathrm{e}_1,m))^2}{2B(\alpha)}$ and adding $\frac{2(1-\alpha)}{B(\alpha)}\nabla(k)$ we have

$$\begin{aligned}&\frac{\alpha(\Upsilon(\mathrm{e}_2, \mathrm{e}_1, m))^2}{2B(\alpha)}\int_0^1 (1-2\eta)\nabla'(m\mathrm{e}_1 + \eta\Upsilon(\mathrm{e}_2, \mathrm{e}_1, m))d\eta + \frac{2(1-\alpha)}{B(\alpha)}\nabla(k)\\ &= -\frac{\alpha\Upsilon(\mathrm{e}_2, \mathrm{e}_1, m)}{B(\alpha)}\frac{\nabla(m\mathrm{e}_1) + \nabla(m\mathrm{e}_1 + \Upsilon(\mathrm{e}_2, \mathrm{e}_1, m))}{2} + \left(\frac{(1-\alpha)}{B(\alpha)}\nabla(k) + \frac{\alpha}{B(\alpha)}\int_{m\mathrm{e}_1}^{k}\nabla(x)dx\right)\\ &+ \left(\frac{(1-\alpha)}{B(\alpha)}\nabla(k) + \frac{\alpha}{B(\alpha)}\int_{k}^{m\mathrm{e}_1+\Upsilon(\mathrm{e}_2,\mathrm{e}_1,m)}\nabla(x)dx\right)\\ &= -\frac{\alpha\Upsilon(\mathrm{e}_2, \mathrm{e}_1, m)}{B(\alpha)}\frac{\nabla(m\mathrm{e}_1) + \nabla(m\mathrm{e}_1 + \Upsilon(\mathrm{e}_2, \mathrm{e}_1, m))}{2} + \left[\left({}^{CF}_{m\mathrm{e}_1}I^{\alpha}\nabla\right)(k) + \left({}^{CF}I^{\alpha}_{m\mathrm{e}_1+\Upsilon(\mathrm{e}_2,\mathrm{e}_1,m)}\nabla\right)(k)\right].\end{aligned}$$

This is the required proof.

Theorem 4.1. *Suppose* ∇ *is defined as in the Lemma 4.1. If* $|\nabla'|$ *is a m-preinvex, then,*

$$\begin{aligned}&\left|-\frac{\nabla(m\mathrm{e}_1) + \nabla(m\mathrm{e}_1 + \Upsilon(\mathrm{e}_2, \mathrm{e}_1, m))}{2} - \frac{2(1-\alpha)}{\alpha\Upsilon(\mathrm{e}_2, \mathrm{e}_1, m)}\nabla(k) + \frac{B(\alpha)}{\alpha\Upsilon(\mathrm{e}_2, \mathrm{e}_1, m)}\left[\left({}^{CF}_{m\mathrm{e}_1}I^{\alpha}\nabla\right)(k) + \left({}^{CF}I^{\alpha}_{m\mathrm{e}_1+\Upsilon(\mathrm{e}_2,\mathrm{e}_1,m)}\nabla\right)(k)\right]\right|\\ &\le \frac{\Upsilon(\mathrm{e}_2, \mathrm{e}_1, m)\left(m|\nabla'(\mathrm{e}_1)| + |\nabla'(\mathrm{e}_2)|\right)}{8},\end{aligned}$$

where $k \in [m\mathrm{e}_1, m\mathrm{e}_1 + \Upsilon(\mathrm{e}_2, \mathrm{e}_1, m)]$.

Proof. Employing lemma 4.1 and $|\nabla'|^q$ satisfying the Definition 2.4 , we have

$$\begin{aligned}&\left|-\frac{\nabla(m\mathrm{e}_1) + \nabla(m\mathrm{e}_1 + \Upsilon(\mathrm{e}_2, \mathrm{e}_1, m))}{2} - \frac{2(1-\alpha)}{\alpha\Upsilon(\mathrm{e}_2, \mathrm{e}_1, m)}\nabla(k) + \frac{B(\alpha)}{\alpha\Upsilon(\mathrm{e}_2, \mathrm{e}_1, m)}\left[\left({}^{CF}_{m\mathrm{e}_1}I^{\alpha}\nabla\right)(k) + \left({}^{CF}I^{\alpha}_{m\mathrm{e}_1+\Upsilon(\mathrm{e}_2,\mathrm{e}_1,m)}\nabla\right)(k)\right]\right|\\ &\le \frac{\Upsilon(\mathrm{e}_2, \mathrm{e}_1, m)}{2}\int_0^1 |1-2\eta||\nabla'(m\mathrm{e}_1 + \eta\Upsilon(\mathrm{e}_2, \mathrm{e}_1, m))|d\eta\\ &\le \frac{\Upsilon(\mathrm{e}_2, \mathrm{e}_1, m)}{2}\int_0^1 |1-2\eta|\left((1-\eta)|m\nabla'(\mathrm{e}_1)| + \eta|\nabla'(\mathrm{e}_2)|\right)d\eta\\ &= \frac{\Upsilon(\mathrm{e}_2, \mathrm{e}_1, m)}{2}\Bigg(\int_0^{1/2}(1-2\eta)\left((1-\eta)|m\nabla'(\mathrm{e}_1)| + \eta|\nabla'(\mathrm{e}_2)|\right)d\eta\\ &+ \int_{1/2}^{1}(1-2\eta)\left((1-\eta)|m\nabla'(\mathrm{e}_1)| + \eta|\nabla'(\mathrm{e}_2)|\right)d\eta\Bigg)\\ &= \frac{\Upsilon(\mathrm{e}_2, \mathrm{e}_1, m)\left(m|\nabla'(\mathrm{e}_1)| + |\nabla'(\mathrm{e}_2)|\right)}{8}.\end{aligned}$$

This is the required proof.

Theorem 4.2. *Suppose ∇ is defined as in the Lemma 4.1. If $|\nabla^{'}|^q$ is a m-preinvex with $p>1$, $\frac{1}{p}+\frac{1}{q}=1$, $\mathtt{e}_1,\mathtt{e}_2\in I$, then*

$$\Big|-\frac{\nabla(m\mathtt{e}_1)+\nabla(m\mathtt{e}_1+\Upsilon(\mathtt{e}_2,\mathtt{e}_1,m))}{2}+\frac{2(1-\alpha)}{\alpha\Upsilon(\mathtt{e}_2,\mathtt{e}_1,m)}\nabla(k)+\frac{B(\alpha)}{\alpha\Upsilon(\mathtt{e}_2,\mathtt{e}_1,m)}\left[\left({}^{CF}_{m\mathtt{e}_1}I^{\alpha}\nabla\right)(k)+\left({}^{CF}I^{\alpha}_{m\mathtt{e}_1+\Upsilon(\mathtt{e}_2,\mathtt{e}_1,m)}\nabla\right)(k)\right]\Big|$$
$$\leq\frac{\Upsilon(\mathtt{e}_2,\mathtt{e}_1,m)}{2}\left(\frac{1}{p+1}\right)^{1/p}\left(\frac{m|\nabla^{'}(\mathtt{e}_1)|^q+|\nabla^{'}(\mathtt{e}_2)|^q}{2}\right)^{1/q},$$

where $k\in[m\mathtt{e}_1,m\mathtt{e}_1+\Upsilon(\mathtt{e}_2,\mathtt{e}_1,m)]$.

Proof. Employing lemma 4.1, Hölder inequality and $|\nabla^{'}|^q$ satisfying the Definition 2.4 , we have

$$\Big|-\frac{\nabla(m\mathtt{e}_1)+\nabla(m\mathtt{e}_1+\Upsilon(\mathtt{e}_2,\mathtt{e}_1,m))}{2}-\frac{2(1-\alpha)}{\alpha\Upsilon(\mathtt{e}_2,\mathtt{e}_1,m)}\nabla(k)+\frac{B(\alpha)}{\alpha\Upsilon(\mathtt{e}_2,\mathtt{e}_1,m)}\left[\left({}^{CF}_{m\mathtt{e}_1}I^{\alpha}\nabla\right)(k)+\left({}^{CF}I^{\alpha}_{m\mathtt{e}_1+\Upsilon(\mathtt{e}_2,\mathtt{e}_1,m)}\nabla\right)(k)\right]\Big|$$
$$\leq\frac{\Upsilon(\mathtt{e}_2,\mathtt{e}_1,m)}{2}\int_0^1|1-2\eta||\nabla^{'}(m\mathtt{e}_1+\eta\Upsilon(\mathtt{e}_2,\mathtt{e}_1,m))|d\eta$$
$$\leq\frac{\Upsilon(\mathtt{e}_2,\mathtt{e}_1,m)}{2}\left(\int_0^1|1-2\eta|^p d\eta\right)^{1/p}\left(\int_0^1|\nabla^{'}(m\mathtt{e}_1+\eta\Upsilon(\mathtt{e}_2,\mathtt{e}_1,m))|^q d\eta\right)^{1/q}$$
$$\leq\frac{\Upsilon(\mathtt{e}_2,\mathtt{e}_1,m)}{2}\left(\frac{1}{p+1}\right)^{1/p}\left(\frac{m|\nabla^{'}(\mathtt{e}_1)|^q+|\nabla^{'}(\mathtt{e}_2)|^q}{2}\right)^{1/q}.$$

So, we have the desired result.

Theorem 4.3. *Suppose ∇ is defined as in the Lemma 4.1. If $|\nabla^{'}|^q$ is a m-preinvex function, then*

$$\Big|-\frac{\nabla(m\mathtt{e}_1)+\nabla(m\mathtt{e}_1+\Upsilon(\mathtt{e}_2,\mathtt{e}_1,m))}{2}-\frac{2(1-\alpha)}{\alpha\Upsilon(\mathtt{e}_2,\mathtt{e}_1,m)}\nabla(k)+\frac{B(\alpha)}{\alpha\Upsilon(\mathtt{e}_2,\mathtt{e}_1,m)}\left[{}^{CF}_{m\mathtt{e}_1}I^{\alpha}\nabla(k)+{}^{CF}I^{\alpha}_{m\mathtt{e}_1+\Upsilon(\mathtt{e}_2,\mathtt{e}_1,m)}\nabla(k)\right]\Big|$$
$$\leq\frac{\Upsilon(\mathtt{e}_2,\mathtt{e}_1,m)}{4}\left(\frac{m|\nabla^{'}(\mathtt{e}_1)|^q+|\nabla^{'}(\mathtt{e}_2)|^q}{2}\right)^{\frac{1}{q}},$$

where $k\in[m\mathtt{e}_1,m\mathtt{e}_1+\Upsilon(\mathtt{e}_2,\mathtt{e}_1,m)]$.

Proof. Employing lemma 4.1, power mean inequality and $|\nabla^{'}|^q$ satisfying the Definition 2.4, we have

$$\Big|-\frac{\nabla(m\mathtt{e}_1)+\nabla(m\mathtt{e}_1+\Upsilon(\mathtt{e}_2,\mathtt{e}_1,m))}{2}-\frac{2(1-\alpha)}{\alpha\Upsilon(\mathtt{e}_2,\mathtt{e}_1,m)}\nabla(k)+\frac{B(\alpha)}{\alpha\Upsilon(\mathtt{e}_2,\mathtt{e}_1,m)}\left[{}^{CF}_{m\mathtt{e}_1}I^{\alpha}\nabla(k)+{}^{CF}I^{\alpha}_{m\mathtt{e}_1+\Upsilon(\mathtt{e}_2,\mathtt{e}_1,m)}\nabla(k)\right]\Big|$$
$$\leq\frac{\Upsilon(\mathtt{e}_2,\mathtt{e}_1,m)}{2}\int_0^1|1-2\eta|\left|\nabla^{'}(m\mathtt{e}_1+\eta\Upsilon(\mathtt{e}_2,\mathtt{e}_1,m))\right|d\eta$$
$$\leq\frac{\Upsilon(\mathtt{e}_2,\mathtt{e}_1,m)}{2}\left(\int_0^1|1-2\eta|d\eta\right)^{1-\frac{1}{q}}\left(|1-2\eta|\left|\nabla^{'}(m\mathtt{e}_1+\eta\Upsilon(\mathtt{e}_2,\mathtt{e}_1,m))\right|^q d\eta\right)^{\frac{1}{q}}$$
$$\leq\frac{\Upsilon(\mathtt{e}_2,\mathtt{e}_1,m)}{2}\left(\frac{1}{2}\right)^{1-\frac{1}{q}}\left(\int_0^1|1-2\eta|\left(m|\nabla^{'}(\mathtt{e}_1)|^q[1-\eta]+|\nabla^{'}(\mathtt{e}_2)|^q\eta\right)d\eta\right)^{\frac{1}{q}}$$
$$\leq\frac{\Upsilon(\mathtt{e}_2,\mathtt{e}_1,m)}{2}\left(\frac{1}{2}\right)^{1-\frac{1}{q}}\left(m|\nabla^{'}(\mathtt{e}_1)|^q\int_0^1|1-2\eta|[1-\eta]d\eta+|\nabla^{'}(\mathtt{e}_2)|^q\int_0^1|1-2\eta|\eta d\eta\right)^{\frac{1}{q}}$$
$$\leq\frac{\Upsilon(\mathtt{e}_2,\mathtt{e}_1,m)}{2}\left(\frac{1}{2}\right)^{1-\frac{1}{q}}\left(\frac{m|\nabla^{'}(\mathtt{e}_1)|^q+|\nabla^{'}(\mathtt{e}_2)|^q}{4}\right)^{\frac{1}{q}}.$$

Further steps lead us to the required proof.

Remark 1. *If we put $\Upsilon(\mathtt{e}_2,\mathtt{e}_1,m)=\mathtt{e}_2-m\mathtt{e}_1$ in the above theorem, then we get*

$$\Big|\frac{\nabla(m\mathtt{e}_1)+\nabla(\mathtt{e}_2)}{2}+\frac{2(1-\alpha)}{\alpha\Upsilon(\mathtt{e}_2,\mathtt{e}_1,m)}\nabla(k)-\frac{B(\alpha)}{\alpha\Upsilon(\mathtt{e}_2,\mathtt{e}_1,m)}\left[{}^{CF}_{m\mathtt{e}_1}I^{\alpha}\nabla(k)+{}^{CF}I^{\alpha}_{\mathtt{e}_2}\nabla(k)\right]\Big|$$
$$\leq\frac{(\mathtt{e}_2-m\mathtt{e}_1)}{4}\left(\frac{|\nabla^{'}(m\mathtt{e}_1)|^q+|\nabla^{'}(\mathtt{e}_2)|^q}{2}\right)^{\frac{1}{q}}.$$

Theorem 4.4. *Suppose ∇ is defined as in the Lemma 4.1. If $|\nabla^{'}|^q$ is a m-preinvex function, then*

$$\Bigg|-\frac{\nabla(m\mathtt{e}_1)+\nabla(\mathtt{e}_1+\Upsilon(\mathtt{e}_2,\mathtt{e}_1,m))}{2}-\frac{2(1-\alpha)}{\alpha\Upsilon(\mathtt{e}_2,\mathtt{e}_1,m)}\nabla(k)+\frac{B(\alpha)}{\alpha\Upsilon(\mathtt{e}_2,\mathtt{e}_1,m)}\left[{}^{CF}_{m\mathtt{e}_1}I^{\alpha}\nabla(k)+^{CF}I^{\alpha}_{m\mathtt{e}_1+\Upsilon(\mathtt{e}_2,\mathtt{e}_1,m)}\nabla(k)\right]\Bigg|$$
$$\leq\frac{\Upsilon(\mathtt{e}_2,\mathtt{e}_1,m)}{4}\left(\frac{1}{p+1}\right)^{\frac{1}{p}}\left[\left(\frac{2m|\nabla^{'}(\mathtt{e}_1)|^q+|\nabla^{'}(\mathtt{e}_2)|^q}{3}\right)^{\frac{1}{q}}+\left(\frac{m|\nabla^{'}(\mathtt{e}_1)|^q+2|\nabla^{'}(\mathtt{e}_2)|^q}{3}\right)^{\frac{1}{q}}\right],$$

where $k\in[m\mathtt{e}_1,m\mathtt{e}_1+\Upsilon(\mathtt{e}_2,\mathtt{e}_1,m)]$.

Proof. Employing lemma 4.1, Holder-Iscan inequality and $|\nabla^{'}|^q$ satisfying the Definition 2.4, we have

$$\Bigg|-\frac{m\nabla(\mathtt{e}_1)+\nabla(m\mathtt{e}_1+\Upsilon(\mathtt{e}_2,\mathtt{e}_1,m))}{2}-\frac{2(1-\alpha)}{\alpha\Upsilon(\mathtt{e}_2,\mathtt{e}_1,m)}\nabla(k)+\frac{B(\alpha)}{\alpha\Upsilon(\mathtt{e}_2,\mathtt{e}_1,m)}\left[{}^{CF}_{m\mathtt{e}_1}I^{\alpha}\nabla(k)+^{CF}I^{\alpha}_{m\mathtt{e}_1+\Upsilon(\mathtt{e}_2,\mathtt{e}_1,m)}\nabla(k)\right]\Bigg|$$
$$\leq\frac{\Upsilon(\mathtt{e}_2,\mathtt{e}_1,m)}{2}\int_0^1|1-2\eta||\nabla^{'}(m\mathtt{e}_1+\eta\Upsilon(\mathtt{e}_2,\mathtt{e}_1,m))|$$
$$\leq\frac{\Upsilon(\mathtt{e}_2,\mathtt{e}_1,m)}{2}\left(\int_0^1(1-\eta)|1-2\eta|^pd\eta\right)^{\frac{1}{p}}\left(\int_0^1(1-\eta)|\nabla^{'}(m\mathtt{e}_1+\eta\Upsilon(\mathtt{e}_2,\mathtt{e}_1,m))|^qd\eta\right)$$
$$+\frac{\Upsilon(\mathtt{e}_2,\mathtt{e}_1,m)}{2}\left(\int_0^1\eta|1-2\eta|^pd\eta\right)^{\frac{1}{p}}\left(\int_0^1\eta|\nabla^{'}(m\mathtt{e}_1+\eta\Upsilon(\mathtt{e}_2,\mathtt{e}_1,m))|^qd\eta\right)$$
$$\leq\frac{\Upsilon(\mathtt{e}_2,\mathtt{e}_1,m)}{2}\left(\frac{1}{2(p+1)}\right)^{\frac{1}{p}}\left(m|\nabla^{'}(\mathtt{e}_1)|^q\int_0^1(1-\eta)(1-\eta)d\eta+|\nabla^{'}(\mathtt{e}_2)|^q\int_0^1(1-\eta)\eta d\eta\right)^{\frac{1}{q}}$$
$$+\frac{\Upsilon(\mathtt{e}_2,\mathtt{e}_1,m)}{2}\left(\frac{1}{2(p+1)}\right)^{\frac{1}{p}}\left(m|\nabla^{'}(\mathtt{e}_1)|^q\int_0^1\eta(1-\eta)d\eta+|\nabla^{'}(\mathtt{e}_2)|^q\int_0^1\eta^2d\eta\right)^{\frac{1}{q}}$$
$$\leq\frac{\Upsilon(\mathtt{e}_2,\mathtt{e}_1,m)}{4}\left(\frac{1}{p+1}\right)^{\frac{1}{p}}\left[\left(\frac{2m|\nabla^{'}(\mathtt{e}_1)|^q+|\nabla^{'}(\mathtt{e}_2)|^q}{3}\right)^{\frac{1}{q}}+\left(\frac{m|\nabla^{'}(\mathtt{e}_1)|^q+2|\nabla^{'}(\mathtt{e}_2)|^q}{3}\right)^{\frac{1}{q}}\right].$$

This is the required proof.

Remark 2. *If we put $\Upsilon(\mathtt{e}_2,\mathtt{e}_1)=\mathtt{e}_2-\mathtt{e}_1$ in the above theorem, then we get*

$$\Bigg|\frac{\nabla(m\mathtt{e}_1)+\nabla(\mathtt{e}_2)}{2}+\frac{2(1-\alpha)}{\alpha(\mathtt{e}_2-m\mathtt{e}_1)}\nabla(k)-\frac{B(\alpha)}{\alpha(\mathtt{e}_2-m\mathtt{e}_1)}\left[{}^{CF}_{m\mathtt{e}_1}I^{\alpha}\nabla(k)+^{CF}I^{\alpha}_{\mathtt{e}_2}\nabla(k)\right]\Bigg|$$
$$\leq\frac{(\mathtt{e}_2-m\mathtt{e}_1)}{4}\left(\frac{1}{p+1}\right)^{\frac{1}{p}}\left[\left(\frac{2m|\nabla^{'}(\mathtt{e}_1)|^q+|\nabla^{'}(\mathtt{e}_2)|^q}{3}\right)^{\frac{1}{q}}+\left(\frac{m|\nabla^{'}(\mathtt{e}_1)|^q+2|\nabla^{'}(\mathtt{e}_2)|^q}{3}\right)^{\frac{1}{q}}\right].$$

Theorem 4.5. *Suppose ∇ is defined as in the Lemma 4.1. If $|\nabla^{'}|^q$ is a m-preinvex function, then*

$$\Bigg|-\frac{\nabla(m\mathtt{e}_1)+\nabla(m\mathtt{e}_1+\Upsilon(\mathtt{e}_2,\mathtt{e}_1,m))}{2}-\frac{2(1-\alpha)}{\alpha\Upsilon(\mathtt{e}_2,\mathtt{e}_1,m)}\nabla(k)+\frac{B(\alpha)}{\alpha\Upsilon(\mathtt{e}_2,\mathtt{e}_1,m)}\left[{}^{CF}_{m\mathtt{e}_1}I^{\alpha}\nabla(k)+^{CF}I^{\alpha}_{m\mathtt{e}_1+\Upsilon(\mathtt{e}_2,\mathtt{e}_1,m)}\nabla(k)\right]\Bigg|$$
$$\leq\frac{\Upsilon(\mathtt{e}_2,\mathtt{e}_1,m)}{8}\left[\left(\frac{3m|\nabla^{'}(\mathtt{e}_1)|^q+|\nabla^{'}(\mathtt{e}_2)|^q}{4}\right)^{\frac{1}{q}}+\left(\frac{m|\nabla^{'}(\mathtt{e}_1)|^q+3|\nabla^{'}(\mathtt{e}_2)|^q}{4}\right)^{\frac{1}{q}}\right],$$

where $k\in[m\mathtt{e}_1,m\mathtt{e}_1+\Upsilon(\mathtt{e}_2,\mathtt{e}_1,m)]$.

Proof. Employing lemma 4.1, improved power mean inequality and $|\nabla'|^q$ satisfying the Definition 2.4, we have

$$\begin{aligned}
&\left|-\frac{\nabla(m\mathsf{e}_1)+\nabla(m\mathsf{e}_1+\Upsilon(\mathsf{e}_2,\mathsf{e}_1,m))}{2}-\frac{2(1-\alpha)}{\alpha\Upsilon(\mathsf{e}_2,\mathsf{e}_1,m)}\nabla(k)+\frac{B(\alpha)}{\alpha\Upsilon(\mathsf{e}_2,\mathsf{e}_1,m)}\left[{}^{CF}_{m\mathsf{e}_1}I^{\alpha}\nabla(k)+{}^{CF}I^{\alpha}_{m\mathsf{e}_1+\Upsilon(\mathsf{e}_2,\mathsf{e}_1,m)}\nabla(k)\right]\right|\\
&\leq\frac{\Upsilon(\mathsf{e}_2,\mathsf{e}_1,m)}{2}\int_0^1|1-2\eta||\nabla'(m\mathsf{e}_1+\eta\Upsilon(\mathsf{e}_2,\mathsf{e}_1,m))|\\
&\leq\frac{\Upsilon(\mathsf{e}_2,\mathsf{e}_1,m)}{2}\left(\int_0^1(1-\eta)|1-2\eta|d\eta\right)^{1-\frac{1}{q}}\left(\int_0^1(1-\eta)|1-2\eta||\nabla'(m\mathsf{e}_1+\eta\Upsilon(\mathsf{e}_2,\mathsf{e}_1,m))|^q d\eta\right)^{\frac{1}{q}}\\
&+\frac{\Upsilon(\mathsf{e}_2,\mathsf{e}_1,m)}{2}\left(\int_0^1\eta|1-2\eta|d\eta\right)^{1-\frac{1}{q}}\left(\int_0^1\eta|1-2\eta||\nabla'(m\mathsf{e}_1+\eta\Upsilon(\mathsf{e}_2,\mathsf{e}_1,m))|^q d\eta\right)^{\frac{1}{q}}\\
&\leq\frac{\Upsilon(\mathsf{e}_2,\mathsf{e}_1,m)}{2}\left(\frac{1}{4}\right)^{1-\frac{1}{q}}\left(m|\nabla'(\mathsf{e}_1)|^q\int_0^1(1-\eta)^2|1-2\eta|d\eta+|\nabla'(\mathsf{e}_2)|^q\int_0^1\eta(1-\eta)|1-2\eta|d\eta\right)^{\frac{1}{q}}\\
&+\frac{\Upsilon(\mathsf{e}_2,\mathsf{e}_1,m)}{2}\left(\frac{1}{4}\right)^{1-\frac{1}{q}}\left(m|\nabla'(\mathsf{e}_1)|^q\int_0^1\eta(1-\eta)|1-2\eta|d\eta+|\nabla'(\mathsf{e}_2)|^q\int_0^1\eta^2|1-2\eta|d\eta\right)^{\frac{1}{q}}\\
&\leq\frac{\Upsilon(\mathsf{e}_2,\mathsf{e}_1,m)}{8}\left[\left(\frac{3m|\nabla'(\mathsf{e}_1)|^q+|\nabla'(\mathsf{e}_2)|^q}{4}\right)^{\frac{1}{q}}+\left(\frac{m|\nabla'(\mathsf{e}_1)|^q+3|\nabla'(\mathsf{e}_2)|^q}{4}\right)^{\frac{1}{q}}\right].
\end{aligned}$$

This is the required proof.

Remark 3. *If we put $\Upsilon(\mathsf{e}_2,\mathsf{e}_1)=\mathsf{e}_2-\mathsf{e}_1$ in the above theorem, then we get*

$$\begin{aligned}
&\left|\frac{\nabla(m\mathsf{e}_1)+\nabla(\mathsf{e}_2)}{2}+\frac{2(1-\alpha)}{\alpha(\mathsf{e}_2-m\mathsf{e}_1)}\nabla(k)-\frac{B(\alpha)}{\alpha(\mathsf{e}_2-m\mathsf{e}_1)}\left[{}^{CF}_{m\mathsf{e}_1}I^{\alpha}\nabla(k)+{}^{CF}I^{\alpha}_{\mathsf{e}_2}\nabla(k)\right]\right|\\
&\leq\frac{(\mathsf{e}_2-m\mathsf{e}_1)}{8}\left[\left(\frac{3m|\nabla'(\mathsf{e}_1)|^q+|\nabla'(\mathsf{e}_2)|^q}{4}\right)^{\frac{1}{q}}+\left(\frac{m|\nabla'(\mathsf{e}_1)|^q+3|\nabla'(\mathsf{e}_2)|^q}{4}\right)^{\frac{1}{q}}\right].
\end{aligned}$$

5. APPLICATIONS

In this portion, we explore some applications pertaining to the above results.

1. The arithmetic mean
$$A=A(\mathsf{e}_1,\mathsf{e}_2)=\frac{\mathsf{e}_1+\mathsf{e}_2}{2},\quad \mathsf{e}_1,\mathsf{e}_2\in R.$$
2. The generalized logarithmic mean
$$L=L_r^r(\mathsf{e}_1,\mathsf{e}_2)=\frac{\mathsf{e}_2{}^{r+1}-\mathsf{e}_1{}^{r+1}}{(r+1)(\mathsf{e}_2-\mathsf{e}_1)},\quad r\in R\setminus\{-1,0\},\quad \mathsf{e}_1,\mathsf{e}_2\in R,\quad \mathsf{e}_1\neq\mathsf{e}_2.$$

We are now demonstrating our results for acquiring some special-means-related inequalities using the results in Section 4.

Proposition 5.1. *Suppose $m\mathsf{e}_1, m\mathsf{e}_1+\Upsilon(\mathsf{e}_2,\mathsf{e}_1,m)\in R^+, m\mathsf{e}_1<m\mathsf{e}_1+\Upsilon(\mathsf{e}_2,\mathsf{e}_1,m)$, then,*

$$\left|-A(m\mathsf{e}_1{}^2,(m\mathsf{e}_1+\Upsilon(\mathsf{e}_2,\mathsf{e}_1,m))^2)+L_2^2(m\mathsf{e}_1,m\mathsf{e}_1+\Upsilon(\mathsf{e}_2,\mathsf{e}_1,m))\right|\leq\frac{\Upsilon(\mathsf{e}_2,\mathsf{e}_1,m)}{4}[m|\mathsf{e}_1|+|\mathsf{e}_2|].$$

Proof. The outcome is obtained if we opt for $\nabla(z)=z^2$ with $\alpha=1$ in Theorem 4.1.

Proposition 5.2. *Let $m\mathsf{e}_1, m\mathsf{e}_1+\Upsilon(\mathsf{e}_2,\mathsf{e}_1,m)\in R^+, m\mathsf{e}_1<m\mathsf{e}_1+\Upsilon(\mathsf{e}_2,\mathsf{e}_1,m)$, then*

$$\left|-A(e^{m\mathsf{e}_1},m\mathsf{e}_1{}^{(m\mathsf{e}_1+\Upsilon(\mathsf{e}_2,\mathsf{e}_1,m))})+L(e^{m\mathsf{e}_1},e^{(m\mathsf{e}_1+\Upsilon(\mathsf{e}_2,\mathsf{e}_1,m))})\right|\leq\frac{\Upsilon(\mathsf{e}_2,\mathsf{e}_1,m)}{8}(e^{m\mathsf{e}_1}+e^{\mathsf{e}_2}).$$

Proof. The outcome is obtained if we opt for $\nabla(z)=e^z$ with $\alpha=1$ in Theorem 4.1.

Proposition 5.3. *Let $m\mathsf{e}_1, m\mathsf{e}_1+\Upsilon(\mathsf{e}_2,\mathsf{e}_1,m)\in R^+, m\mathsf{e}_1<m\mathsf{e}_1+\Upsilon(\mathsf{e}_2,\mathsf{e}_1,m)$, then*

$$\left|-A(m\mathsf{e}_1{}^n,(m\mathsf{e}_1+\Upsilon(\mathsf{e}_2,\mathsf{e}_1,m))^n)+L_n^n(m\mathsf{e}_1,m\mathsf{e}_1+\Upsilon(\mathsf{e}_2,\mathsf{e}_1,m))\right|\leq\frac{n\Upsilon(\mathsf{e}_2,\mathsf{e}_1,m)}{8}\left[m|\mathsf{e}_1{}^{n-1}|+|\mathsf{e}_2{}^{n-1}|\right].$$

Proof. The outcome is obtained if we opt for $\nabla(z)=z^2$ with $\alpha=1$.

6. CONCLUSIONS

The research on fractional integral inequalities has captured the keen interest of numerous scientists from a broad spectrum of subject areas because of prospective uses for fractional calculus. When contrasted with convex functions, generalizations and computations made using preinvexity yield better and sharper bounds. A novel version of the H-H inequality and its generalizations involving a fractional integral operator in the Caputo-Fabrizio sense were examined and explored in this work.

Author Contributions

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Conflicts of Interest

The authors declare that there are no conflicts of interest.

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All data relevant to the study are included in the article.

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