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Review Article

Hermite-Hadamard Type Inequality Via Caputo-Fabrizio Fractional Integral Operators: A Review Note

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ABSTRACT

The idea of convexity is mainly employed in the realm of fractional calculus to address a number of issues in pure and practical research. The purpose of this review note is to demonstrate Hermite-Hadamard (H-H) inequalities via Caputo-Fabrizio fractional integral operators associated with various topic of convexity.

1. INTRODUCTION

The background of convex functions is deep and distinguished. Convexity theory has an extensive record that dates back to the late nineteenth century. In the applied sciences, convex theory gives us the right rules and methods to concentrate on a wide variety of issues. Convexity theory has played great behavior in the advancement of several fields in modern applied sciences mathematics, including economics, engineering, optimization, and financial mathematics.

Recently, integral inequalities have been extensively recognized as having aided in the advancement of numerous mathematical concepts as well as other scientific fields. The primary goal of fractional calculus is to build mathematical models. Fractional inequalities are essential for proving the uniqueness of solutions for particular fractional partial differential equations. They also provide upper and lower bounds for fractional boundary value problem solutions.

In recent years, one of the most interesting areas of mathematical inquiry has been the extension of well-known inequalities in the framework of fractional operators. Integrating the concept of mathematical inequalities into the mode of Caputo-Fabrizio fractional operators—which are especially captivating due to their properties and widespread application, is the aim of this work. Subjects such as statistics, operation research, probability, mathematical economics, mathematical programming, game theory and variational methods have all shown a great deal of interest in the problem outside of mathematics. For the literature, see the reference [1–5].

Convexity's current state is summed up in this review, which also improves the reader's comprehension by going over new research findings. Our aim is to present a thorough and in-depth review that highlights advancements in the field by incorporating as many results as possible. For the sake of conciseness, long proofs are left out, but readers are referred to the pertinent articles for these specifics.

For an assortment of reasons, a review paper is essential for higher learning and scientific investigations. Review papers

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synthesize a variety of studies to give a broad picture of the momentous state of research on a given subject. This makes it easier for readers to understand what is established, what has been demonstrated, and what is still up for debate. Review papers draw attention to important patterns, recurring discoveries, and new concepts. This will be particularly helpful to scientists looking for developments or imperfections in the compilation of existing literature. One of the most valuable factors in a review paper is capacity to identify imperfections or areas that have not been fully explored. This can serve as inspiration for subsequent investigations by indicating out areas that require more inquiry or innovative solutions. Review articles might offer perspectives on understudied subtopics or methodologies, encouraging new lines of inquiry and innovative thinking in the field. For scientists who only recently started out in a field, a review paper can serve as a road map of the key studies, approaches, and arguments. Review papers assists viewers perceive the relevance and novelty of fresh research by setting it in context by examining how it relates to earlier studies. A review paper frequently displays the development of a particular subject over time. It offers important insights into the development of scientific understanding by looking at how ideas, concepts, or innovations have changed over time. Review papers frequently receive a lot of citations, because they offer a useful resource for anyone working on the same topic. When citing a review paper, scientists have fast and simple access to a large number of sources.

1.1 Convex Function

Definition 1. *[6] Suppose $\mathcal{Q} : \mathbb{I} \to \mathbb{R}$ be a function, then*

$$\mathcal{Q}\left(\wp\pi_1 + (1-\wp)\pi_2\right) \le \wp\mathcal{Q}\left(\pi_1\right) + (1-\wp)\mathcal{Q}\left(\pi_2\right), \tag{1}$$

is said to be convex, if $\forall\ \pi_1, \pi_2 \in \mathbb{I}$ and $\wp \in [0,1]$.

1.2 Hermite-Hadamard Inequality

One of the fundamental ideas in the applied sciences, which have many fascinating applications, is inequality. In this way, several extensions of the well-known H-H inequalities have been extended for generalizations on convex functions. Hermite [7] and Hadamard [8], respectively, introduced and studied this inequality in 1893. This inequality states that:
If $\mathcal{Q} : \mathbb{I} \to \mathbb{R}$ is CF in $\mathbb{I}$ for $\ell_1, \ell_2 \in \mathbb{I}$ and $\ell_1 < \ell_2$, then

$$\mathcal{Q}\left(\frac{\pi_1+\pi_2}{2}\right) \le \frac{1}{\pi_2-\pi_1}\int_{\pi_1}^{\pi_2}\mathcal{Q}(\wp)d\wp \le \frac{\mathcal{Q}(\pi_1)+\mathcal{Q}(\pi_2)}{2}. \tag{2}$$

2. H-H TYPE INEQUALITY VIA CAPUTO-FABRIZIO FRACTIONAL INTEGRAL OPERATOR

Definition 2 ([9]). *Let $\mathcal{Q} \in H^1(\pi_1,\pi_2) = \{y \in L^2(\pi_1,\pi_2) : y' \in L^2(\pi_1,\pi_2)\}$, $\pi_1 < \pi_2$, $\alpha \in [0,1]$, then the definition of the left fractional integral in the sense of C-F becomes*

$$^{CF}I^{\alpha}_{\pi_1}\mathcal{Q}(t) = \frac{1-\alpha}{B(\alpha)}\mathcal{Q}(t) + \frac{\alpha}{B(\alpha)}\int_{\pi_1}^{t}\mathcal{Q}(x)dx,$$

the right fractional integral

$$^{CF}I^{\alpha}_{\pi_2}\mathcal{Q}(t) = \frac{1-\alpha}{B(\alpha)}\mathcal{Q}(t) + \frac{\alpha}{B(\alpha)}\int_{t}^{b}\mathcal{Q}(x)dx,$$

where $B(\alpha) > 0$ is a normalization function satisfying $B(0) = B(1) = 1$.

The operator has a number of significant advantages over conventional definitions like the Riemann-Liouville or Caputo operators. Its non-singular kernel—a power-law is replaced by an exponential function is one of its most significant features. This excludes singularities at the origin and renders it more appropriate for the modeling of physical processes with finite memory and smooth initial-time behavior. The Caputo-Fabrizio operator is often applied by scientists in order to simulate real-world systems, like viscoelastic materials, anomalous diffusion, biological systems, or epidemiological models, in which memory effects are present but do not have a strong singularity. Besides this, its definition enhances the stability and applicability of numerical solutions. The CF operator has become increasingly popular due to its advantages in computation and physical interpretability. Its use in enhancing the realism of mathematical models has been at the center of several recent works, especially in fields where fractional derivatives are applied, but traditional singular kernels are seen as physically impractical. With the aim of finding a balance between mathematical rigor, physical relevance, and numerical efficiency, researchers prefer using the Caputo-Fabrizio operator.

2.1 H-H-type inequalities for convex function via CFFIO

Gürbüz [10] presented the following H-H-type inequality and its generalizations for convexity via CFFIO.

Theorem 2.1 ([10]). *Let $\mathcal{Q} : [\pi_1, \pi_2] \subseteq \mathbb{R} \to \mathbb{R}$ be convex function on $[\pi_1, \pi_2]$ and $\mathcal{Q} \in L_1[\pi_1, \pi_2]$. If $\alpha \in [0, 1]$, then*

$$\mathcal{Q}\Big(\frac{\pi_1 + \pi_2}{2}\Big) \leq \frac{B(\alpha)}{\alpha(\pi_2 - \pi_1)}\Big[{}^{CF}I^{\alpha}_{\pi_1}\mathcal{Q}(k) + {}^{CF}I^{\alpha}_{\pi_2}\mathcal{Q}(k) - \frac{2(1-\alpha)}{B(\alpha)}\mathcal{Q}(k)\Big] \leq \frac{\mathcal{Q}(\pi_1) + \mathcal{Q}(\pi_2)}{2}.$$

Theorem 2.2 ([10]). *Suppose that $\mathcal{Q}$ is defined in Theorem 2.1 and differentiable. If $\mathcal{Q}' \in L_1[\pi_1, \pi_2]$ and $\alpha \in [0, 1]$, then*

$$\begin{aligned} &\frac{\mathcal{Q}(\pi_1) + \mathcal{Q}(\pi_2)}{2} + \frac{2(1-\alpha)}{\alpha(\pi_2 - \pi_1)}\mathcal{Q}(k) - \frac{B(\alpha)}{\alpha(\pi_2 - \pi_1)}\Big[{}^{CF}I^{\alpha}_{\pi_1}\mathcal{Q}(k) + {}^{CF}I^{\alpha}_{\pi_2}\mathcal{Q}(k)\Big] \\ \leq\ & \frac{(\pi_2 - \pi_1)[|\mathcal{Q}'(\pi_1)| + |\mathcal{Q}'(\pi_2)|]}{8}. \end{aligned}$$

Tariq [11] presented the following H-H-type inequality and its extensions for convex function via CFFIO.

Theorem 2.3 ([11]). *Let $\mathcal{Q}$ is defined in Theorem 2.1. If $|\mathcal{Q}'|$ is a convexity, then*

$$\begin{aligned} &\Big|\frac{B(\alpha)}{\alpha(\pi_2 - \pi_1)}\Big[{}^{CF}I^{\alpha}_{\frac{\pi_1+\pi_2}{2}-}\mathcal{Q}(\pi_1) + {}^{CF}I^{\alpha}_{\frac{\pi_1+\pi_2}{2}+}\mathcal{Q}(\pi_2)\Big] - \mathcal{Q}\Big(\frac{\pi_1 + \pi_2}{2}\Big)\Big| \\ \leq\ & \frac{\pi_2 - \pi_1}{4}\Big\{\frac{|\mathcal{Q}'(\pi_1)| + |\mathcal{Q}'(\pi_2)|}{2}\Big\} + \frac{1-\alpha}{\alpha(\pi_2 - \pi_1)}\Big[\mathcal{Q}(\pi_1) + \mathcal{Q}(\pi_2)\Big]. \end{aligned}$$

Theorem 2.4 ([11]). *Let $\mathcal{Q}$ is defined in Theorem 2.1, $q > 1$ and $\frac{1}{p} + \frac{1}{q} = 1$. If $|\mathcal{Q}'|^q$ is a convexity mapping, then*

$$\begin{aligned} &\Big|\frac{B(\alpha)}{\alpha(\pi_2 - \pi_1)}\Big[{}^{CF}I^{\alpha}_{\frac{\pi_1+\pi_2}{2}-}\mathcal{Q}(\pi_1) + {}^{CF}I^{\alpha}_{\frac{\pi_1+\pi_2}{2}+}\mathcal{Q}(\pi_2)\Big] - \mathcal{Q}\Big(\frac{\pi_1 + \pi_2}{2}\Big) \\ &- \frac{1-\alpha}{\alpha(\pi_2 - \pi_1)}\Big[\mathcal{Q}(\pi_1) + \mathcal{Q}(\pi_2)\Big]\Big| \\ \leq\ & \frac{\pi_2 - \pi_1}{4}\Big\{\Big(\frac{1}{p+1}\Big)^{\frac{1}{p}}\Big[\Big(\frac{|\mathcal{Q}'(\pi_1)|^q}{4} + \frac{3|\mathcal{Q}'(\pi_2)|^q}{4}\Big)^{\frac{1}{q}} + \Big(\frac{3|\mathcal{Q}'(\pi_1)|^q}{4} + \frac{|\mathcal{Q}'(\pi_2)|^q}{4}\Big)^{\frac{1}{q}}\Big]\Big\}. \end{aligned}$$

Theorem 2.5 ([11]). *Let $\alpha \in [0, 1]$ and $\mathcal{Q} : \mathbb{I} \to \mathbb{R}$ is a convexity mapping such that $\pi_1, \pi_2 \in \mathbb{I}$ and $\mathcal{Q} \in L_1[\pi_1, \pi_2]$. Then*

$$\begin{aligned} \mathcal{Q}\Big(\frac{\pi_1 + \pi_2}{2}\Big) &\leq \frac{B(\alpha)}{\alpha(\pi_2 - \pi_1)}\Big[{}^{CF}I^{\alpha}_{\frac{\pi_1+\pi_2}{2}-}\mathcal{Q}(\pi_1) + {}^{CF}I^{\alpha}_{\frac{\pi_1+\pi_2}{2}+}\mathcal{Q}(\pi_2) - \frac{1-\alpha}{B(\alpha)}[\mathcal{Q}(\pi_1) + \mathcal{Q}(\pi_2)]\Big] \\ &\leq \frac{\mathcal{Q}(\pi_1) + \mathcal{Q}(\pi_2)}{2}. \end{aligned}$$

Theorem 2.6 ([11]). *Let mapping $\mathcal{Q} : \mathbb{I} = [\pi_1, \pi_2] \to \mathbb{R}$ is a differentiable on $\mathbb{I}$ such that $\pi_1, \pi_2 \in \mathbb{I}$ with $\pi_1 < \pi_2$ and $\mathcal{Q} \in L_1[\pi_1, \pi_2]$. If mapping $|\mathcal{Q}'|$ is a convex, then*

$$\begin{aligned} &\Big|\frac{B(\alpha)}{\alpha(\pi_2 - \pi_1)}\Big[{}^{CF}I^{\alpha}_{\frac{\pi_1+\pi_2}{2}-}\mathcal{Q}(\pi_1) + {}^{CF}I^{\alpha}_{\frac{\pi_1+\pi_2}{2}+}\mathcal{Q}(\pi_2) - \mathcal{Q}\Big(\frac{\pi_1 + \pi_2}{2}\Big)\Big| \\ \leq\ & \frac{\pi_2 - \pi_1}{4}\Big[\frac{\mathcal{Q}'(\pi_1) + \mathcal{Q}'(\pi_2)}{2}\Big] + \frac{1-\alpha}{\alpha(\pi_2 - \pi_1)}[\mathcal{Q}(\pi_1) + \mathcal{Q}(\pi_2)]. \end{aligned}$$

Theorem 2.7 ([11]). *Let mapping $\mathcal{Q} : \mathbb{I} = [\pi_1, \pi_2] \to \mathbb{R}$ is a differentiable on $\mathbb{I}$ such that $\pi_1, \pi_2 \in \mathbb{I}$ with $\pi_1 < \pi_2$ and $\mathcal{Q} \in L_1[\pi_1, \pi_2]$. If $|\mathcal{Q}'|^q$, $q > 1$ is a convexity, then, for $\alpha \in [0, 1]$, $\frac{1}{p} + \frac{1}{q} = 1$, we have*

$$\begin{aligned} &\Big|\frac{B(\alpha)}{\alpha(\pi_2 - \pi_1)}\Big[{}^{CF}I^{\alpha}_{\frac{\pi_1+\pi_2}{2}-}\mathcal{Q}(\pi_1) + {}^{CF}I^{\alpha}_{\frac{\pi_1+\pi_2}{2}+}\mathcal{Q}(\pi_2) - \mathcal{Q}\Big(\frac{\pi_1 + \pi_2}{2}\Big) \\ &- \frac{1-\alpha}{\alpha(\pi_2 - \pi_1)}[\mathcal{Q}(\pi_1) + \mathcal{Q}(\pi_2)]\Big| \\ \leq\ & \frac{\pi_2 - \pi_1}{4}\Big\{\Big(\frac{1}{p+1}\Big)^{\frac{1}{p}}\Big[\Big(\frac{|\mathcal{Q}'(\pi_1)|^q}{4} + \frac{3|\mathcal{Q}'(\pi_2)|^q}{4}\Big)^{\frac{1}{q}} + \Big(\frac{3|\mathcal{Q}'(\pi_1)|^q}{4} + \frac{|\mathcal{Q}'(\pi_2)|^q}{4}\Big)^{\frac{1}{q}}\Big]\Big\}. \end{aligned}$$

2.2 H-H-type inequalities for s-convex functions via CFFIO

Abbasi [12] presented the H-H-type inequality and its refinements for s- convexity via CFFIO.

Theorem 2.8 ([12]). *Let $\mathcal{Q} : [\pi_1, \pi_2] \to \mathbb{R}$ be s-convex on $[\pi_1, \pi_2]$ for $s \in (0, 1)$ and $\mathcal{Q} \in L_1[\pi_1, \pi_2]$. If $\alpha \in [0, 1]$, then*

$$\begin{aligned} 2^{s-1}\mathcal{Q}\Big(\frac{\pi_1+\pi_2}{2}\Big) &\leq \frac{B(\alpha)}{\alpha(\pi_2-\pi_1)}\Big[({}^{CF}I^{\alpha}_{\pi_1}\mathcal{Q})(k) + ({}^{CF}I^{\alpha}_{\pi_2}\mathcal{Q})(k) - \frac{2(1-\alpha)}{B(\alpha)}\mathcal{Q}(k)\Big] \\ &\leq \frac{\mathcal{Q}(\pi_1)+\mathcal{Q}(\pi_2)}{2}, \end{aligned}$$

where $k \in [\pi_1, \pi_2]$.

Theorem 2.9 ([12]). *Assume that function $\mathcal{Q} : [\pi_1, \pi_2] \to \mathbb{R}$ is differentiable on (π_1, π_2) and $|\mathcal{Q}'|$ is an s-convexity on $[\pi_1, \pi_2]$ for $s \in (0, 1)$ and $\mathcal{Q}' \in L_1[\pi_1, \pi_2]$. If $\alpha \in [0, 1]$, then*

$$\begin{aligned} &\left|\frac{\mathcal{Q}(\pi_1)+\mathcal{Q}(\pi_2)}{2} - \frac{B(\alpha)}{\alpha(\pi_2-\pi_1)}\Big[({}^{CF}I^{\alpha}_{\pi_1}\mathcal{Q})(k) + ({}^{CF}I^{\alpha}_{\pi_2}\mathcal{Q})(k) - \frac{2(1-\alpha)}{B(\alpha)}\mathcal{Q}(k)\Big]\right| \\ \leq\ & \frac{\pi_2-\pi_1}{2}\Big(\frac{2^{-s}+s}{(s+1)(s+2)}\Big)(|\mathcal{Q}'(x)| + |\mathcal{Q}'(\pi_2)|), \end{aligned}$$

where $k \in [\pi_1, \pi_2]$.

2.3 H-H-type inequalities for n-polynomial s-type convex function via CFFIO

Definition 3 ([13]). *Assume that $n \in \mathbb{N}$, $\pi_1, \pi_2 \in \mathrm{T}$ with $\wp, s \in [0, 1]$, then an inequality of the form*

$$\mathcal{Q}(\ell\pi_1 + (1-\ell)\pi_2) \leq \frac{1}{n}\sum_{\wp=1}^{n}\Big[1-(s(1-\ell))^{\wp}\Big]\mathcal{Q}(\pi_1) + \frac{1}{n}\sum_{\wp=1}^{n}\Big[1-(s\ell)^{\wp}\Big]\mathcal{Q}(\pi_2).$$

is said be n-polynomial s-type convex function.

Tariq [14] presented the following H-H-type inequality for Definition 3 via CFFIO.

Theorem 2.10 ([14]). *Let $\mathcal{Q} : \mathrm{T} \to \mathbb{R}$ be n-polynomial s-type convexity on T with $\pi_1 < \pi_2$ and $\pi_1, \pi_2 \in \mathrm{T}$. If $\mathcal{Q}$ is Lebesgue integrable mapping on $[\pi_1, \pi_2]$ then,*

$$\begin{aligned} \frac{2^{-1}n}{\sum_{\wp=1}^{n}\left[1-\left(\frac{s}{2}\right)^{\wp}\right]}\mathcal{Q}\left(\frac{\pi_1+\pi_2}{2}\right) &\leq \frac{B(\alpha)}{\alpha(\pi_2-\pi_1)}\left[{}^{CF}\mathrm{I}^{\alpha}_{\pi_1}\mathcal{Q}(r) + {}^{CF}\mathrm{I}^{\alpha}_{\pi_2}\mathcal{Q}(r) - \frac{2(1-\alpha)}{B(\alpha)}\mathcal{Q}(r)\right] \\ &\leq \frac{\mathcal{Q}(\pi_1)+\mathcal{Q}(\pi_2)}{n}\sum_{\wp=1}^{n}\frac{p+1-s^p}{p+1}, \end{aligned}$$

where $\alpha \in [0, 1]$, $s \in [0, 1]$, $r \in [0, 1]$.

2.4 H-H-type inequalities for (s, m)-convex functions via CFFIO

Tariq [15] presented the following H-H-type inequality and its refinements for (s, m)-convexity via CFFIO.

Theorem 2.11 ([15]). *Assume that $\mathcal{Q} : [\pi_1, \pi_2] \to \mathbb{R}$ is integrable and (s, m)-convexity on $[\pi_1, \pi_2]$. Then*

$$\begin{aligned} 2^{s}\mathcal{Q}\Big(\frac{\pi_1+m\pi_2}{2}\Big) &\leq \frac{B(\alpha)}{\alpha(m\pi_2-\pi_1)}\Big(\Big[({}^{CF}I^{\alpha}_{\pi_1}\mathcal{Q})(x) + ({}^{CF}I^{\alpha}_{m\pi_2}\mathcal{Q})(x)\Big] \\ &\quad + m^2\Big[({}^{CF}I^{\alpha}_{\pi_1/m}\mathcal{Q})(x) + ({}^{CF}I^{\alpha}_{\pi_2}\mathcal{Q})(x)\Big] - \frac{2(1+m^2)(1-\alpha)}{B(\alpha)}\mathcal{Q}(x)\Big) \\ &\leq \Big[\frac{\mathcal{Q}(\pi_1)+m\mathcal{Q}(\pi_2)}{s+1} + m\frac{\mathcal{Q}(\pi_2)+m\mathcal{Q}\Big(\frac{\pi_1}{m^2}\Big)}{s+1}\Big], \end{aligned}$$

where $\alpha > 0$.

Theorem 2.12 ([15])**.** *Let $\mathcal{Q} : [\pi_1, m\pi_2] \to \mathbb{R}$ be differentiable mapping on $(\pi_1, m\pi_2)$. If $\mathcal{Q}'$ is integrable and (s, m)-convexity on $[\pi_1, m\pi_2]$, then*

$$\begin{aligned}&\frac{\mathcal{Q}(\pi_1) + \mathcal{Q}(m\pi_2)}{2} - \frac{B(\alpha)}{\alpha(m\pi_2 - \pi_1)}\Big[({}^{CF}I^{\alpha}_{\pi_1}\mathcal{Q})(x) + ({}^{CF}I^{\alpha}_{m\pi_2}\mathcal{Q})(x)\Big] + \frac{2(1-\alpha)}{\alpha(m\pi_2 - \pi_1)}\mathcal{Q}(x)\\ \leq\ & \frac{m\pi_2 - \pi_1}{2}[m\mathcal{Q}'(\pi_2) - \mathcal{Q}'(\pi_1)]\frac{s}{(s+1)(s+2)}.\end{aligned}$$

Theorem 2.13 ([15])**.** *Assume that $\mathcal{Q} : [\pi_1, m\pi_2] \to \mathbb{R}$ is a twice differentiable mapping on $(\pi_1, m\pi_2)$. If $\mathcal{Q}''$ is integrable and (s, m)-convexity on $[\pi_1, m\pi_2]$, then*

$$\begin{aligned}&\frac{\mathcal{Q}(\pi_1) + \mathcal{Q}(m\pi_2)}{2} - \frac{B(\alpha)}{\alpha(m\pi_2 - \pi_1)}\Big[({}^{CF}I^{\alpha}_{\pi_1}\mathcal{Q})(x) + ({}^{CF}I^{\alpha}_{m\pi_2}\mathcal{Q})(x)\Big] + \frac{2(1-\alpha)}{\alpha(m\pi_2 - \pi_1)}\mathcal{Q}(x)\\ \leq\ & \frac{(m\pi_2 - \pi_1)^2}{2}\frac{[\mathcal{Q}''(\pi_1) + m\mathcal{Q}''(\pi_2)]}{(s+3)(s+2)}.\end{aligned}$$

2.5 H-H type inequalities involving n-polynomial generalized m-convex involving Raina's function pertaining to CFFIO

Definition 4. *(see [16]) Let $\pi_1, \pi_2 \in X$, $m \in (0, 1]$, $\mathfrak{e} \in [0, 1]$ and $\mathfrak{n} \in \mathbb{N}$. An inequality*

$$\mathcal{Q}(m\pi_1 + \mathfrak{e}\mathcal{R}^{\varrho}_{\epsilon,\theta}(\pi_2 - m\pi_1)) \leq \frac{m}{\mathfrak{n}}\sum_{\mathfrak{z}=1}^{\mathfrak{n}}(1 - \mathfrak{e}^{\mathfrak{z}})\,\mathcal{Q}\,(\pi_1) + \frac{1}{\mathfrak{n}}\sum_{\mathfrak{z}=1}^{\mathfrak{n}}(1 - (1-\mathfrak{e})^{\mathfrak{z}})\mathcal{Q}\,(\pi_2)\,, \tag{3}$$

is said to be n-polynomial generalized m-convex involving Raina's function i.e., PG_mCRF.

Tariq [16] presented the following H-H-type inequality and its generalizations for PG_mCRF via CFFIO.

Theorem 2.14. *[16] Suppose $\mathcal{Q} : [m\pi_1, m\pi_1 + \mathcal{R}^{\varrho}_{\epsilon,\theta}(\pi_2 - m\pi_1)] \to \mathbb{R}$ is PG_mCRF and satisfies extended condition-A. Then*

$$\begin{aligned}\frac{1}{2}\Big(\frac{\mathfrak{n}}{\mathfrak{n} + 2^{-\mathfrak{n}} - 1}\Big)\mathcal{Q}\Big(m\pi_1 + \frac{1}{2}\mathcal{R}^{\varrho}_{\epsilon,\theta}(\pi_2 - m\pi_1)\Big) &\leq \frac{1}{\mathcal{R}^{\varrho}_{\epsilon,\theta}(\pi_2 - m\pi_1)}\int_{m\pi_1}^{m\pi_1 + \mathcal{R}^{\varrho}_{\epsilon,\theta}(\pi_2 - m\pi_1)}\mathcal{Q}(x)dx\\ &\leq \frac{m\mathcal{Q}(\pi_1) + \mathcal{Q}(\pi_2)}{\mathfrak{n}}\sum_{\mathfrak{z}=1}^{\mathfrak{n}}\frac{\mathfrak{z}}{\mathfrak{z}+1}.\end{aligned} \tag{4}$$

Theorem 2.15. *[16] The statement of this theorem is stated in Theorem 2.14, then*

$$\begin{aligned}&\frac{2^{-1}\mathfrak{n}}{\mathfrak{n} + 2^{-\mathfrak{n}} - 1}\mathcal{Q}\left(\frac{2m\pi_1 + \mathcal{R}^{\varrho}_{\epsilon,\theta}(\pi_2 - m\pi_1)}{2}\right)\\ &\leq \frac{\mathcal{B}(\sigma)}{\sigma\mathcal{R}^{\varrho}_{\epsilon,\theta}(\pi_2 - m\pi_1)}\left[\left({}^{\mathcal{CF}}_{m\pi_1}\mathrm{I}^{\sigma}\mathcal{Q}\right)(\mathfrak{v}) + \left({}^{\mathcal{CF}}\mathrm{I}^{\sigma}_{m\pi_1 + \mathcal{R}^{\varrho}_{\epsilon,\theta}(\pi_2 - m\pi_1)}\mathcal{Q}\right)(\mathfrak{v}) - \frac{2(1-\sigma)}{\mathcal{B}(\sigma)}\mathcal{Q}(\mathfrak{v})\right]\\ &\leq \frac{m\mathcal{Q}(\pi_1) + \mathcal{Q}(\pi_2)}{\mathfrak{n}}\sum_{\mathfrak{z}=1}^{\mathfrak{n}}\frac{\mathfrak{z}}{\mathfrak{z}+1}.\end{aligned}$$

(A_5) Let $\mathcal{Q} : \mathbb{I} = [m\pi_1, m\pi_1 + \mathcal{R}^{\varrho}_{\epsilon,\theta}(\pi_2 - m\pi_1)] \to \mathbb{R}$ be a differentiable function on $\mathbb{I}^{\circ}$, $m\pi_1 < m\pi_1 + \mathcal{R}^{\varrho}_{\epsilon,\theta}(\pi_2 - m\pi_1)$ and $|\mathcal{Q}'|$ is PG_mCRF on $[m\pi_1, m\pi_1 + \mathcal{R}^{\varrho}_{\epsilon,\theta}(\pi_2 - m\pi_1)]$.

Theorem 2.16. *[16] With consideration in (A_5). If $\mathcal{Q}' \in \mathcal{L}[m\pi_1, m\pi_1 + \mathcal{R}^{\varrho}_{\epsilon,\theta}(\pi_2 - m\pi_1)]$, then*

$$\begin{aligned}&\Bigg|-\frac{\mathcal{Q}(m\pi_1) + \mathcal{Q}(m\pi_1 + \mathcal{R}^{\varrho}_{\epsilon,\theta}(\pi_2 - m\pi_1))}{2} - \frac{2(1-\sigma)}{\sigma\mathcal{R}^{\varrho}_{\epsilon,\theta}(\pi_2 - m\pi_1)}\mathcal{Q}(\mathfrak{v})\\ &+ \frac{\mathcal{B}(\sigma)}{\sigma\mathcal{R}^{\varrho}_{\epsilon,\theta}(\pi_2 - m\pi_1)}\left[\left({}^{\mathcal{CF}}_{m\pi_1}\mathrm{I}^{\sigma}\mathcal{Q}\right)(\mathfrak{v}) + \left({}^{\mathcal{CF}}\mathrm{I}^{\sigma}_{m\pi_1 + \mathcal{R}^{\varrho}_{\epsilon,\theta}(\pi_2 - m\pi_1)}\mathcal{Q}\right)(\mathfrak{v})\right]\Bigg|\\ &\leq \frac{\mathcal{R}^{\varrho}_{\epsilon,\theta}(\pi_2 - m\pi_1)}{\mathfrak{n}}\sum_{\mathfrak{z}=1}^{\mathfrak{n}}\left[\frac{(\mathfrak{z}^2 + \mathfrak{z} + 2)2^{\mathfrak{z}}\mathfrak{z} - 2}{(\mathfrak{z}+1)(\mathfrak{z}+2)2^{\mathfrak{z}+1}}\right]\left(\frac{m|\mathcal{Q}'(\pi_1)| + |\mathcal{Q}'(\pi_2)|}{2}\right),\end{aligned} \tag{5}$$

Theorem 2.17. *[16] With consideration in (A_5). If $\mathcal{Q}' \in \mathcal{L}[m\pi_1, m\pi_1 + \mathcal{R}^{\varrho}_{\epsilon,\theta}(\pi_2 - m\pi_1)]$, then*

$$\Bigg| - \frac{\mathcal{Q}(m\pi_1) + \mathcal{Q}(m\pi_1 + \mathcal{R}^{\varrho}_{\epsilon,\theta}(\pi_2 - m\pi_1))}{2} - \frac{2(1-\sigma)}{\sigma \mathcal{R}^{\varrho}_{\epsilon,\theta}(\pi_2 - m\pi_1)}\mathcal{Q}(\mathfrak{v}) + \frac{\mathcal{B}(\sigma)}{\sigma \mathcal{R}^{\varrho}_{\epsilon,\theta}(\pi_2 - m\pi_1)}\left[\left({}^{\mathcal{CF}}_{m\pi_1}\mathrm{I}^{\sigma}\mathcal{Q}\right)(\mathfrak{v}) + \left({}^{\mathcal{CF}}\mathrm{I}^{\sigma}_{m\pi_1 + \mathcal{R}^{\varrho}_{\epsilon,\theta}(\pi_2 - m\pi_1)}\mathcal{Q}\right)(\mathfrak{v})\right]\Bigg| \le \frac{\mathcal{R}^{\varrho}_{\epsilon,\theta}(\pi_2 - m\pi_1)}{2}\left(\frac{1}{p+1}\right)^{\frac{1}{p}}\left(\frac{2}{\mathfrak{n}}\sum_{\mathfrak{z}=1}^{\mathfrak{n}}\frac{\mathfrak{z}}{\mathfrak{z}+1}\right)^{\frac{1}{q}}\left(\frac{m|\mathcal{Q}'(\pi_1)|^q + |\mathcal{Q}'(\pi_2)|^q}{2}\right)^{\frac{1}{q}}. \tag{6}$$

Theorem 2.18. *[16] With consideration in (A_5) and assume that $\mathcal{Q}' \in \mathcal{L}[m\pi_1, m\pi_1 + \mathcal{R}^{\varrho}_{\epsilon,\theta}(\pi_2 - m\pi_1)]$. If $|\mathcal{Q}'|^q$ is PG_mCRF, then*

$$\Bigg| - \frac{\mathcal{Q}(m\pi_1) + \mathcal{Q}(m\pi_1 + \mathcal{R}^{\varrho}_{\epsilon,\theta}(\pi_2 - m\pi_1))}{2} - \frac{2(1-\sigma)}{\sigma \mathcal{R}^{\varrho}_{\epsilon,\theta}(\pi_2 - m\pi_1)}\mathcal{Q}(\mathfrak{v}) + \frac{\mathcal{B}(\sigma)}{\sigma \mathcal{R}^{\varrho}_{\epsilon,\theta}(\pi_2 - m\pi_1)}\left[\left({}^{\mathcal{CF}}_{m\pi_1}\mathrm{I}^{\sigma}\mathcal{Q}\right)(\mathfrak{v}) + \left({}^{\mathcal{CF}}\mathrm{I}^{\sigma}_{m\pi_1 + \mathcal{R}^{\varrho}_{\epsilon,\theta}(\pi_2 - m\pi_1)}\mathcal{Q}\right)(\mathfrak{v})\right]\Bigg| \le \frac{\mathcal{R}^{\varrho}_{\epsilon,\theta}(\pi_2 - m\pi_1)}{2}\left(\frac{1}{2}\right)^{1-\frac{1}{q}}\left(\frac{1}{\mathfrak{n}}\sum_{\mathfrak{z}=1}^{\mathfrak{n}}\frac{(\mathfrak{z}^2+\mathfrak{z}+2)2^{\mathfrak{z}}-2}{(\mathfrak{z}+1)(\mathfrak{z}+2)2^{\mathfrak{z}+1}}\right)^{\frac{1}{q}}\left(\frac{m|\mathcal{Q}'(\pi_1)|^q + |\mathcal{Q}'(\pi_2)|^q}{2}\right)^{\frac{1}{q}}. \tag{7}$$

Theorem 2.19. *[16] With consideration in (A_5), $q > 1$, $\frac{1}{p} + \frac{1}{q} = 1$ and suppose that $\mathcal{Q}' \in \mathcal{L}[m\pi_1, m\pi_1 + \mathcal{R}^{\varrho}_{\epsilon,\theta}(\pi_2 - m\pi_1)]$. If $|\mathcal{Q}'|^q$ is PG_mCRF, then*

$$\begin{aligned}&\Bigg| - \frac{\mathcal{Q}(m\pi_1) + \mathcal{Q}(m\pi_1 + \mathcal{R}^{\varrho}_{\epsilon,\theta}(\pi_2 - m\pi_1))}{2} - \frac{2(1-\sigma)}{\sigma \mathcal{R}^{\varrho}_{\epsilon,\theta}(\pi_2 - m\pi_1)}\mathcal{Q}(\mathfrak{v}) \\ &+ \frac{\mathcal{B}(\sigma)}{\sigma \mathcal{R}^{\varrho}_{\epsilon,\theta}(\pi_2 - m\pi_1)}\left[\left({}^{\mathcal{CF}}_{m\pi_1}\mathrm{I}^{\sigma}\mathcal{Q}\right)(\mathfrak{v}) + \left({}^{\mathcal{CF}}\mathrm{I}^{\sigma}_{m\pi_1 + \mathcal{R}^{\varrho}_{\epsilon,\theta}(\pi_2 - m\pi_1)}\mathcal{Q}\right)(\mathfrak{v})\right]\Bigg| \\ &\le \frac{\mathcal{R}^{\varrho}_{\epsilon,\theta}(\pi_2 - m\pi_1)}{2}\left(\frac{1}{2(p+1)}\right)^{\frac{1}{p}}\left(\frac{m|\mathcal{Q}'(\pi_1)|^q}{\mathfrak{n}}\sum_{\mathfrak{z}=1}^{\mathfrak{n}}\frac{\mathfrak{z}(\mathfrak{z}+3)}{2(\mathfrak{z}+1)(\mathfrak{z}+2)} + \frac{|\mathcal{Q}'(\pi_2)|^q}{\mathfrak{n}}\sum_{\mathfrak{z}=1}^{\mathfrak{n}}\frac{\mathfrak{z}}{2(\mathfrak{z}+2)}\right)^{\frac{1}{q}} \\ &+ \frac{\mathcal{R}^{\varrho}_{\epsilon,\theta}(\pi_2 - m\pi_1)}{2}\left(\frac{1}{2(p+1)}\right)^{\frac{1}{p}}\left(\frac{|\mathcal{Q}'(\pi_2)|^q}{\mathfrak{n}}\sum_{\mathfrak{z}=1}^{\mathfrak{n}}\frac{\mathfrak{z}(\mathfrak{z}+3)}{2(\mathfrak{z}+1)(\mathfrak{z}+2)} + \frac{|\mathcal{Q}'(\pi_1)|^q}{\mathfrak{n}}\sum_{\mathfrak{z}=1}^{\mathfrak{n}}\frac{\mathfrak{z}}{2(\mathfrak{z}+2)}\right)^{\frac{1}{q}}.\end{aligned}$$

Theorem 2.20. *[16] With consideration in (A_5), $q \ge 1$ and suppose that $\mathcal{Q}' \in \mathcal{L}[m\pi_1, m\pi_1 + \mathcal{R}^{\varrho}_{\epsilon,\theta}(\pi_2 - m\pi_1)]$. If $|\mathcal{Q}'|^q$ is PG_mCRF, then*

$$\begin{aligned}&\Bigg| - \frac{\mathcal{Q}(m\pi_1) + \mathcal{Q}(m\pi_1 + \mathcal{R}^{\varrho}_{\epsilon,\theta}(\pi_2 - m\pi_1))}{2} - \frac{2(1-\sigma)}{\sigma \mathcal{R}^{\varrho}_{\epsilon,\theta}(\pi_2 - m\pi_1)}\mathcal{Q}(\mathfrak{v}) \\ &+ \frac{\mathcal{B}(\sigma)}{\sigma \mathcal{R}^{\varrho}_{\epsilon,\theta}(\pi_2 - m\pi_1)}\left[\left({}^{\mathcal{CF}}_{m\pi_1}\mathrm{I}^{\sigma}\mathcal{Q}\right)(\mathfrak{v}) + \left({}^{\mathcal{CF}}\mathrm{I}^{\sigma}_{m\pi_1 + \mathcal{R}^{\varrho}_{\epsilon,\theta}(\pi_2 - m\pi_1)}\mathcal{Q}\right)(\mathfrak{v})\right]\Bigg| \\ &\le \frac{\mathcal{R}^{\varrho}_{\epsilon,\theta}(\pi_2 - m\pi_1)}{2}\left(\frac{1}{2}\right)^{2-\frac{2}{q}} \\ &\times\left(\frac{m|\mathcal{Q}'(\pi_1)|^q}{\mathfrak{n}}\sum_{\mathfrak{z}=1}^{\mathfrak{n}}\frac{(\mathfrak{z}+5)[(\mathfrak{z}^2+\mathfrak{z}+2)2^{\mathfrak{z}}-2]}{2^{\mathfrak{z}+2}(\mathfrak{z}+1)(\mathfrak{z}+2)(\mathfrak{z}+3)} + \frac{|\mathcal{Q}'(\pi_2)|^q}{\mathfrak{n}}\sum_{\mathfrak{z}=1}^{\mathfrak{n}}\frac{(\mathfrak{z}^2+\mathfrak{z}+2)2^{\mathfrak{z}}-2}{2^{\mathfrak{z}+2}(\mathfrak{z}+2)(\mathfrak{z}+3)}\right)^{\frac{1}{q}} \\ &+ \frac{\mathcal{R}^{\varrho}_{\epsilon,\theta}(\pi_2 - m\pi_1)}{2}\left(\frac{1}{2}\right)^{2-\frac{2}{q}} \\ &\times\left(\frac{m|\mathcal{Q}'(\pi_1)|^q}{\mathfrak{n}}\sum_{\mathfrak{z}=1}^{\mathfrak{n}}\frac{(\mathfrak{z}^2+\mathfrak{z}+2)2^{\mathfrak{z}}-2}{2^{\mathfrak{z}+2}(\mathfrak{z}+2)(\mathfrak{z}+3)} + \frac{|\mathcal{Q}'(\pi_2)|^q}{\mathfrak{n}}\sum_{\mathfrak{z}=1}^{\mathfrak{n}}\frac{(\mathfrak{z}+5)[(\mathfrak{z}^2+\mathfrak{z}+2)2^{\mathfrak{z}}-2]}{2^{\mathfrak{z}+2}(\mathfrak{z}+1)(\mathfrak{z}+2)(\mathfrak{z}+3)}\right)^{\frac{1}{q}}.\end{aligned}$$

2.6 H-H-type inequalities involving preinvex functions via CFFIO

Tariq [17] presented the following H-H-type inequality and its refinements for preinvexity via CFFIO.

Theorem 2.21 ([17]). *Suppose* $\mathcal{Q} : [\pi_1, \pi_1 + \eta(\pi_2, \pi_1)] \to (0, \infty)$ *be a preinvex mapping and* $\mathcal{Q} \in L_1[\pi_1, \pi_1 + \eta(\pi_2, \pi_1)]$. *If* $\alpha \in [0, 1]$, *then we have:*

$$\begin{aligned}\mathcal{Q}\Big(\frac{2\pi_1 + \eta(\pi_2, \pi_1)}{2}\Big) &\leq \frac{B(\alpha)}{\alpha\eta(\pi_2, \pi_1)}\Big[{}^{CF}I^{\alpha}_{\pi_1}\mathcal{Q}(k) + {}^{CF}I^{\alpha}_{(\pi_1+\eta(\pi_2,\pi_1))}\mathcal{Q}(k) - \frac{2(1-\alpha)}{B(\alpha)}\mathcal{Q}(k)\Big] \\ &\leq \frac{\mathcal{Q}(\pi_1) + \mathcal{Q}(\pi_2)}{2},\end{aligned}$$

where $k \in [\pi_1, \pi_1 + \eta(\pi_2, \pi_1)]$.

(A_6) Assume that $\mathcal{Q} : I = [\pi_1, \pi_1 + \eta(\pi_2, \pi_1)] \to (0, \infty)$ is a differentiable mapping on I° and $|\mathcal{Q}'|$ be a preinvex function on $[\pi_1, \pi_1 + \eta(\pi_2, \pi_1)]$.

Theorem 2.22 ([17]). *With consideration in* (A_6). *If* $\mathcal{Q}' \in L_1[\pi_1, \pi_1 + \eta(\pi_2, \pi_1)]$, *then*

$$\begin{aligned}&\Bigg| - \frac{\mathcal{Q}(\pi_1) + \mathcal{Q}(\pi_1 + \eta(\pi_2, \pi_1)}{2} - \frac{2(1-\alpha)}{B(\alpha)}\mathcal{Q}(k) \\ &+ \frac{B(\alpha)}{\alpha\eta(\pi_2, \pi_1)}\Big[{}^{CF}I^{\alpha}_{\pi_1}\mathcal{Q}(k) + {}^{CF}I^{\alpha}_{(\pi_1+\eta(\pi_2,\pi_1))}\mathcal{Q}(k)\Big]\Bigg| \\ \leq\ & \frac{\eta(\pi_2, \pi_1)(|\mathcal{Q}'(\pi_1)| + |\mathcal{Q}'(\pi_2)|)}{8}.\end{aligned}$$

Theorem 2.23 ([17]). *With consideration in* (A_6). *If* $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$ *and* $\mathcal{Q}' \in L_1[\pi_1, \pi_1 + \eta(\pi_2, \pi_1)]$, *then*

$$\begin{aligned}&\Bigg| - \frac{\mathcal{Q}(\pi_1) + \mathcal{Q}(\pi_1 + \eta(\pi_2, \pi_1)}{2} - \frac{2(1-\alpha)}{B(\alpha)}\mathcal{Q}(k) \\ &+ \frac{B(\alpha)}{\alpha\eta(\pi_2, \pi_1)}\Big[{}^{CF}I^{\alpha}_{\pi_1}\mathcal{Q}(k) + {}^{CF}I^{\alpha}_{(\pi_1+\eta(\pi_2,\pi_1))}\mathcal{Q}(k)\Big]\Bigg| \\ \leq\ & \frac{\eta(\pi_2, \pi_1)}{2}\Big(\frac{1}{p+1}\Big)^{\frac{1}{p}}\Big(\frac{(|\mathcal{Q}'(\pi_1)|^q + |\mathcal{Q}'(\pi_2)|^q)}{2}\Big)^{\frac{1}{q}}.\end{aligned}$$

2.7 H-H type inequalities involving n-polynomial preinvexity via CFFIO

Definition 5 ([18]). *A function* $\mathcal{Q} : \mathbb{I} \to \mathbb{R}$ *is an n-polynomial preinvex, if*

$$\mathcal{Q}(\pi_1 + \mu\xi(\pi_2, \pi_1)) \leq \frac{1}{n}\sum_{s=1}^{n}[1 - \mu^s]\,\mathcal{Q}(\pi_1) + \frac{1}{n}\sum_{s=1}^{n}[1 - (1-\mu)^s]\,\mathcal{Q}(\pi_2), \tag{8}$$

$\forall\ \pi_1, \pi_2 \in \mathbb{I}$, $n \in \mathbb{N}$. *and* $\mu \in [0, 1]$.

Tariq [19] presented the following H-H-type inequality and its generalization for Definition 5 via CFFIO.

Theorem 2.24. *[19] Suppose* $\mathcal{Q} : [\pi_1, \pi_1 + \xi(\pi_2, \pi_1)] \to \mathbb{R}$ *satisfies Definition 5 and condition-C, then*

$$\begin{aligned}&\frac{1}{2}\Big(\frac{n}{n + 2^{-n} - 1}\Big)\mathcal{Q}\Big(\pi_1 + \frac{1}{2}\xi(\pi_2, \pi_1)\Big) \\ &\leq \frac{1}{\xi(\pi_2, \pi_1)}\int_{\pi_1}^{\pi_1+\xi(\pi_2,\pi_1)}\mathcal{Q}(x)dx \leq \frac{\mathcal{Q}(\pi_1) + \mathcal{Q}(\pi_2)}{n}\sum_{s=1}^{n}\frac{s}{s+1}.\end{aligned} \tag{9}$$

Theorem 2.25. *[19] Suppose* $\mathcal{Q} : \mathbb{I} = [\pi_1, \pi_1 + \xi(\pi_2, \pi_1)] \to \mathbb{R}$ *satisfies Definition 5, then*

$$\begin{aligned}&\frac{2^{-1}n}{n + 2^{-n} - 1}\mathcal{Q}\left(\frac{2\pi_1 + \xi(\pi_2, \pi_1)}{2}\right) \\ &\leq \frac{\xi(\lambda)}{\lambda\xi(\pi_2, \pi_1)}\left[\left({}^{\mathcal{CF}}_{\pi_1}\mathrm{I}^{\lambda}\mathcal{Q}\right)(k) + \left({}^{\mathcal{CF}}\mathrm{I}^{\lambda}_{\pi_1+\xi(\pi_2,\pi_1)}\mathcal{Q}\right)(k) - \frac{2(1-\lambda)}{\xi(\lambda)}\mathcal{Q}(k)\right] \\ &\leq \frac{\mathcal{Q}(\pi_1) + \mathcal{Q}(\pi_2)}{n}\sum_{s=1}^{n}\frac{s}{s+1},\end{aligned}$$

where $\lambda \in [0,1]$ *and* $k \in [\pi_1, \pi_1 + \xi(\pi_2,\pi_1)]$.

(A_7) Let $\mathcal{Q} : \mathbb{I} = [\pi_1, \pi_1 + \xi(\pi_2,\pi_1)] \to \mathbb{R}$ is a differentiable function on $\mathbb{I}^\circ$, $\pi_1 < \pi_1 + \xi(\pi_2,\pi_1)$ and $|\mathcal{Q}'|$ is a generalized preinvex function on $[\pi_1, \pi_1 + \xi(\pi_2,\pi_1)]$.

Theorem 2.26. *[19] With consideration in* (A_7)*. If* $\mathcal{Q}' \in \mathcal{L}[\pi_1, \pi_1 + \xi(\pi_2,\pi_1)]$*, then*

$$\begin{aligned}
&\Bigg| -\frac{\mathcal{Q}(\pi_1) + \mathcal{Q}(\pi_1 + \xi(\pi_2,\pi_1))}{2} - \frac{2(1-\lambda)}{\lambda\xi(\pi_2,\pi_1)}\mathcal{Q}(k)\\
&+ \frac{\xi(\lambda)}{\lambda\xi(\pi_2,\pi_1)}\left[\left({}^{\mathcal{CF}}_{\pi_1}\mathrm{I}^{\lambda}\mathcal{Q}\right)(k) + \left({}^{\mathcal{CF}}\mathrm{I}^{\lambda}_{\pi_1+\xi(\pi_2,\pi_1)}\mathcal{Q}\right)(k)\right]\Bigg|\\
&\le \frac{\xi(\pi_2,\pi_1)}{n}\sum_{\mathsf{s}=1}^{n}\left[\frac{(\mathsf{s}^2+\mathsf{s}+2)2^{\mathsf{s}}\mathsf{s}-2}{(\mathsf{s}+1)(\mathsf{s}+2)2^{\mathsf{s}+1}}\right]\left(\frac{|\mathcal{Q}'(\pi_1)| + |\mathcal{Q}'(\pi_2)|}{2}\right).
\end{aligned} \tag{10}$$

Theorem 2.27. *[19] With consideration in* (A_7)*. If* $\mathcal{Q}' \in \mathcal{L}[\pi_1, \pi_1 + \xi(\pi_2,\pi_1)]$*, then*

$$\begin{aligned}
&\Bigg| -\frac{\mathcal{Q}(\pi_1) + \mathcal{Q}(\pi_1 + \xi(\pi_2,\pi_1))}{2} - \frac{2(1-\lambda)}{\lambda\xi(\pi_2,\pi_1)}\mathcal{Q}(k)\\
&+ \frac{\xi(\lambda)}{\lambda\xi(\pi_2,\pi_1)}\left[\left({}^{\mathcal{CF}}_{\pi_1}\mathrm{I}^{\lambda}\mathcal{Q}\right)(k) + \left({}^{\mathcal{CF}}\mathrm{I}^{\lambda}_{\pi_1+\xi(\pi_2,\pi_1)}\mathcal{Q}\right)(k)\right]\Bigg|\\
&\le \frac{\xi(\pi_2,\pi_1)}{2}\left(\frac{1}{p+1}\right)^{\frac{1}{p}}\left(\frac{2}{n}\sum_{\mathsf{s}=1}^{n}\frac{\mathsf{s}}{\mathsf{s}+1}\right)^{\frac{1}{q}}\left(\frac{|\mathcal{Q}'(\pi_1)|^q + |\mathcal{Q}'(\pi_2)|^q}{2}\right)^{\frac{1}{q}}.
\end{aligned} \tag{11}$$

Theorem 2.28. *[19] With consideration in* (A_7)*. If* $|\mathcal{Q}'|^q$ *is a generalized preinvex, then*

$$\begin{aligned}
&\Bigg| -\frac{\mathcal{Q}(\pi_1) + \mathcal{Q}(\pi_1 + \xi(\pi_2,\pi_1))}{2} - \frac{2(1-\lambda)}{\lambda\xi(\pi_2,\pi_1)}\mathcal{Q}(k)\\
&+ \frac{\xi(\lambda)}{\lambda\xi(\pi_2,\pi_1)}\left[\left({}^{\mathcal{CF}}_{\pi_1}\mathrm{I}^{\lambda}\mathcal{Q}\right)(k) + \left({}^{\mathcal{CF}}\mathrm{I}^{\lambda}_{\pi_1+\xi(\pi_2,\pi_1)}\mathcal{Q}\right)(k)\right]\Bigg|\\
&\le \frac{\xi(\pi_2,\pi_1)}{2}\left(\frac{1}{2}\right)^{1-\frac{1}{q}}\left(\frac{1}{n}\sum_{\mathsf{s}=1}^{n}\frac{(\mathsf{s}^2+\mathsf{s}+2)2^{\mathsf{s}}-2}{(\mathsf{s}+1)(\mathsf{s}+2)2^{\mathsf{s}+1}}\right)^{\frac{1}{q}}\left(\frac{|\mathcal{Q}'(\pi_1)|^q + |\mathcal{Q}'(\pi_2)|^q}{2}\right)^{\frac{1}{q}}.
\end{aligned} \tag{12}$$

Theorem 2.29. *[19] With consideration in* (A_7)*. Let* $q > 1$, $p^{-1} + q^{-1} = 1$*, and* $\mathcal{Q}' \in \mathcal{L}[\pi_1, \pi_1 + \xi(\pi_2,\pi_1)]$*. If* $|\mathcal{Q}'|^q$ *is a generalized preinvex, then*

$$\begin{aligned}
&\Bigg| -\frac{\mathcal{Q}(\pi_1) + \mathcal{Q}(\pi_1 + \xi(\pi_2,\pi_1))}{2} - \frac{2(1-\lambda)}{\lambda\xi(\pi_2,\pi_1)}\mathcal{Q}(k)\\
&+ \frac{\xi(\lambda)}{\lambda\xi(\pi_2,\pi_1)}\left[\left({}^{\mathcal{CF}}_{\pi_1}\mathrm{I}^{\lambda}\mathcal{Q}\right)(k) + \left({}^{\mathcal{CF}}\mathrm{I}^{\lambda}_{\pi_1+\xi(\pi_2,\pi_1)}\mathcal{Q}\right)(k)\right]\Bigg|\\
&\le \frac{\xi(\pi_2,\pi_1)}{2}\left(\frac{1}{2(p+1)}\right)^{\frac{1}{p}}\left(\frac{|\mathcal{Q}'(\pi_1)|^q}{n}\frac{(n+n^2)}{(2(2+n))} + \frac{|\mathcal{Q}'(\pi_2)|^q}{n}\sum_{\mathsf{s}=1}^{n}\frac{\mathsf{s}}{2(\mathsf{s}+2)}\right)^{\frac{1}{q}}\\
&+ \frac{\xi(\pi_2,\pi_1)}{2}\left(\frac{1}{2(p+1)}\right)^{\frac{1}{p}}\left(\frac{|\mathcal{Q}'(\pi_2)|^q}{n}\frac{(n+n^2)}{(2(2+n))} + \frac{|\mathcal{Q}'(\pi_1)|^q}{n}\sum_{\mathsf{s}=1}^{n}\frac{\mathsf{s}}{2(\mathsf{s}+2)}\right)^{\frac{1}{q}}.
\end{aligned}$$

Theorem 2.30. *[19] With consideration in* (A_7)*. Let* $q \geq 1$*, and* $|\mathcal{Q}'|^q$ *is an generalized preinvex, then*

$$\Bigg| -\frac{\mathcal{Q}(\pi_1)+\mathcal{Q}(\pi_1+\xi(\pi_2,\pi_1))}{2} - \frac{2(1-\lambda)}{\lambda\xi(\pi_2,\pi_1)}\mathcal{Q}(k)$$
$$+\frac{\xi(\lambda)}{\lambda\xi(\pi_2,\pi_1)}\left[\left({}^{\mathcal{CF}}_{\pi_1}\mathrm{I}^{\lambda}\mathcal{Q}\right)(k)+\left({}^{\mathcal{CF}}\mathrm{I}^{\lambda}_{\pi_1+\xi(\pi_2,\pi_1)}\mathcal{Q}\right)(k)\right]\Bigg|$$
$$\leq \frac{\xi(\pi_2,\pi_1)}{2}\left(\frac{1}{2}\right)^{2-\frac{2}{q}}$$
$$\times\left(\frac{|\mathcal{Q}'(\pi_1)|^q}{n}\sum_{\mathrm{s}=1}^{n}\frac{(\mathrm{s}+5)[(\mathrm{s}^2+\mathrm{s}+2)2^{\mathrm{s}}-2]}{2^{\mathrm{s}+2}(\mathrm{s}+1)(\mathrm{s}+2)(\mathrm{s}+3)}+\frac{|\mathcal{Q}'(\pi_2)|^q}{n}\sum_{\mathrm{s}=1}^{n}\frac{(\mathrm{s}^2+\mathrm{s}+2)2^{\mathrm{s}}-2}{2^{\mathrm{s}+2}(\mathrm{s}+2)(\mathrm{s}+3)}\right)^{\frac{1}{q}}$$
$$+\frac{\xi(\pi_2,\pi_1)}{2}\left(\frac{1}{2}\right)^{2-\frac{2}{q}}$$
$$\times\left(\frac{|\mathcal{Q}'(\pi_1)|^q}{n}\sum_{\mathrm{s}=1}^{n}\frac{(\mathrm{s}^2+\mathrm{s}+2)2^{\mathrm{s}}-2}{2^{\mathrm{s}+2}(\mathrm{s}+2)(\mathrm{s}+3)}+\frac{|\mathcal{Q}'(\pi_2)|^q}{n}\sum_{\mathrm{s}=1}^{n}\frac{(\mathrm{s}+5)[(\mathrm{s}^2+\mathrm{s}+2)2^{\mathrm{s}}-2]}{2^{\mathrm{s}+2}(\mathrm{s}+1)(\mathrm{s}+2)(\mathrm{s}+3)}\right)^{\frac{1}{q}}.$$

3. CONCLUSIONS

The intention of this review article was to offer a meaningful and novel analysis of the H-H inequalities associated with CFFIO. This review was drawn up with attention to the practical and theoretical importance of the H-H inequalities. Before we come up with new findings, we believe that this current overview will inspire and offer a place for mathematicians studying H-H inequality to learn about earlier research on the subject. The review's implications for further research are positive, and we anticipate that it will lead to a significant increase in research.

Conflicts Of Interest

"The authors declare no conflicts of interest."

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