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Research Article

Solving Obstacle Problems using Optimal Homotopy Asymptotic Method

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ABSTRACT

Differential equations play a vital role in explaining real-world phenomena across science and engineering, from fluid motion and population growth to the mechanics of bridges. In particular, the cantilever bridge problem can be framed as a homogeneous obstacle problem, which illustrates the practical importance of such mathematical models. Although obstacle problems have been studied extensively, many existing approaches leave gaps that limit their effectiveness. In this work, we apply the Optimal Homotopy Asymptotic Method (OHAM) to derive exact solutions for several obstacle problems. Our findings not only demonstrate the accuracy and efficiency of this approach but also highlight interesting structural symmetries, as revealed through graphical results that provide deeper insight into the nature of these problems.

1. Introduction

1.1. Background and Literature Overview

The remarkable progress in science and engineering during the past century has been driven largely by the use of mathematics as a tool for modeling and solving real-world problems. Across different eras, mathematics has served as the foundation of scientific advancement-helping researchers to formulate models, generalize principles, and design techniques that provide solutions to both theoretical and applied problems. With the advent of high-speed digital computers, increasingly complex systems could be explored, many of which are naturally expressed in terms of ordinary differential equations (ODEs) and partial differential equations (PDEs).

One of the most influential mathematical frameworks in this context is variational inequality theory (VIT). This theory provides a powerful way of reformulating physical and engineering problems into differential equation models. Its applications span elasticity, solid and fluid mechanics, porous media flow, transportation systems, optimal control, and structural analysis, among others [1]. A central foundation of VIT is the treatment of unilateral problems, such as the interaction between elastic bodies and rigid obstacles. Kikuchi and Oden [1] demonstrated that the equilibrium of an elastic body in contact with a rigid foundation can be rigorously analyzed using variational inequalities. Stampacchia [2] also contributed to the theoretical basis of variational inequalities, while Sobolev spaces and Hilbert space formulations [3,4] provide the essential functional analytic tools for their rigorous treatment.

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As a result, a large body of literature has emerged concerning boundary value problems (BVPs) associated with unilateral, obstacle, and contact conditions. In this work, we focus specifically on obstacle problems of contact type, and first review the general formulation.

1.2.General Variational Inequality

Let H be a real Hilbert Space equipped with an inner product $\langle \cdot,\cdot \rangle$ and norm $\|\cdot\|$. We define:

- K: a closed convex set in H,
- f: a linear continuous functional on H,
- T, g: nonlinear operators on H.

The general variational inequality problem seeks $u \in H$ such that $g(u) \in K$ and

$$\langle \mathfrak{T}\mathfrak{u}, \mathfrak{g}(\mathfrak{v}) - \mathfrak{g}(\mathfrak{u}) \rangle \geq \langle \mathfrak{f}, \mathfrak{g}(\mathfrak{v}) - \mathfrak{g}(\mathfrak{u}) \rangle, \forall\, \mathfrak{v} \in H \tag{1'}$$

The existence of solutions for (1′) is well established in the literature [2–4]. This inequality provides the foundation for studying various classes of obstacle problems, which will be addressed in the following sections.

1.3. Higher-Order Obstacle Problems

A general form of higher-order obstacle problems can be written as:

$$\mathfrak{u}^{(\mathfrak{n})}(\mathfrak{x}) = \begin{cases} \mathfrak{P}(\mathfrak{f}(\mathfrak{x}), \mathfrak{g}(\mathfrak{x}), \mathfrak{u}(\mathfrak{x}), \mathfrak{r}), \mathfrak{a} \leq \mathfrak{x} < \mathfrak{c} \\ \mathfrak{Q}(\mathfrak{f}(\mathfrak{x}), \mathfrak{g}(\mathfrak{x}), \mathfrak{u}(\mathfrak{x}), \mathfrak{r}), \mathfrak{c} \leq \mathfrak{x} < \mathfrak{d} \\ \mathfrak{R}(\mathfrak{f}(\mathfrak{x}), \mathfrak{g}(\mathfrak{x}), \mathfrak{u}(\mathfrak{x}), \mathfrak{r}), \mathfrak{d} \leq \mathfrak{x} < \mathfrak{b} \end{cases} \tag{1}$$

With boundary conditions

$$\begin{aligned} &\mathfrak{u}^{(\mathfrak{k})}(\mathfrak{a}) = \mathfrak{u}^{(\mathfrak{k})}(\mathfrak{b}) = \alpha_{\mathfrak{k}}\mathfrak{k} = 0,1,2,\ldots,\mathfrak{n}-1 \\ &\mathfrak{u}^{(\mathfrak{k})}(\mathfrak{c}) = \mathfrak{u}^{(\mathfrak{k})}(\mathfrak{d}) = \gamma_{\mathfrak{k}}\mathfrak{k} = 0,1,2,\ldots,\mathfrak{n}-1. \end{aligned} \tag{2}$$

Here, $\mathfrak{P}, \mathfrak{Q}, \mathfrak{R}: \mathbb{R}^4 \to \mathbb{R}$ and the functions $\mathfrak{u}^{(\mathfrak{k})}(\mathfrak{x}), \mathfrak{k} = 0,1,2,\ldots,\mathfrak{n}-1$are continuous on $\mathfrak{c}$ and $\mathfrak{d}$. Exact analytic solutions of such problems are generally difficult, and thus numerical approaches have become indispensable [5-9].

1.4.Numerical Methods for Obstacle Problems

The study of obstacle problems has expanded rapidly due to its applications in mechanics, elasticity, and fluid flow. Over the last few decades, considerable progress has been achieved in developing numerical methods capable of handling their complexity.

- Second-order problems: Early work employed spline-based approaches. Cubic spline techniques were introduced in [5], while parametric cubic splines [6] and sextic spline functions [7] were later developed, providing improved accuracy compared to finite difference or collocation methods. Taylor–Galerkin B-spline finite element methods [8] and B-spline-based schemes [9] further enriched the numerical toolkit.
- Third-order problems: Al-Said [5] and Khan & Aziz [6] explored cubic and quintic spline methods, while Caglar et al. [9] applied B-spline techniques. Additional contributions to third-order obstacle problems include spline-based approaches by Al-Said & Noor [10] and others.
- Fourth-order problems: Several approaches have been reported, including collocation methods with cubic and quintic splines [11-13], parametric spline methods [12], adaptive finite element approaches [14], and nonpolynomial spline techniques [15, 16]. Noor & Khalifa [11, 12] provided numerical frameworks for unilateral and odd-order problems, highlighting the flexibility of spline collocation methods.

- Higher-order problems: Fifth- and tenth-order boundary value problems have been addressed using nonpolynomial splines [15,16]. Usmani [17] contributed to the analysis of elastic plate bending under fire conditions, while Tonti [18] developed a general variational formulation for nonlinear problems.

Collectively, these contributions demonstrate the effectiveness of spline-based methods, finite element approaches, and penalty techniques in providing accurate approximations for obstacle problems where closed-form solutions are not attainable.

In conclusion, variational inequality theory offers a unifying framework for studying obstacle and contact problems in mathematical physics and engineering. Although exact solutions are rarely possible, significant progress has been achieved through numerical methods, particularly spline-based and finite element techniques. These developments-from early spline schemes [5,6] to adaptive FEM [14] and beyond-illustrate both the mathematical richness of obstacle problems and their importance in applied sciences.

2. Formulation of Obstacle Problem

For better understanding, we will start with some 2nd order obstacle BVPs for finding the solution $\mathfrak{u}(\mathfrak{x})$ given that

$$\left.\begin{aligned} -\mathfrak{u}''(\mathfrak{x}) &\geq \mathfrak{f}(\mathfrak{x}) && \text{on} \\ \mathfrak{u}(\mathfrak{x}) &\geq \Psi(\mathfrak{x}) && \text{on} \\ [\mathfrak{u}''(\mathfrak{x})+\mathfrak{f}(\mathfrak{x})][\mathfrak{u}(\mathfrak{x})-\Psi(\mathfrak{x})] &= 0 && \text{on} \end{aligned}\right\} \begin{aligned} \Omega &= [0,1] \\ \Omega &= [0,1] \\ \Omega &= [0,1] \end{aligned} \qquad (5)$$
$$\mathfrak{u}(0)=0, \mathfrak{u}'(0)=0, \mathfrak{u}'(1)=0$$

Where $\Psi(\mathfrak{x})$ and $\mathfrak{f}(\mathfrak{x})$ represent the elastic obstacle function and continuous function. The obstacle equations (5) define the static form of the adjustable string, dragged eventually and stretched upon a flexible impediment $\Psi(\mathfrak{x})$.To look at the problem in the light of VIT, we have to proceed in the following way. We construct the set $\mathfrak{K}$ as

$$\mathfrak{K} = \{\nu : \nu \in H_0^2(\Omega) \wedge \nu \geq \Psi on \Omega\}$$

Which represents a closed convex subset in $H_0^2(\Omega)$, where $H_0^2(\Omega)$ is Sobolove Space and Hilbert Space as well [3,4]. The associated power operational $I[\nu]$ by impediment equation (5) through the Tonti approach [18] is:

$$\begin{aligned} I[\nu] &= \int_0^1 (-\nu''(\mathfrak{x}) - 2\mathfrak{f}(\mathfrak{x}))\nu(\mathfrak{x})d\mathfrak{x}\,\forall \nu \in \mathfrak{K} \\ &= -[\nu'(\mathfrak{x})\nu(\mathfrak{x})]|_0^1 + \int_0^1 \nu'(\mathfrak{x})\nu'(\mathfrak{x})d\mathfrak{x} - 2\int_0^1 \mathfrak{f}(\mathfrak{x})\nu(\mathfrak{x})d\mathfrak{x} \\ &= -[\nu(1)\nu'(1)] + [\nu(0)\nu'(0)] + \int_0^1 \nu'(\mathfrak{x})\nu'(\mathfrak{x})d\mathfrak{x} - 2\int_0^1 \mathfrak{f}(\mathfrak{x})\nu(\mathfrak{x})d\mathfrak{x} \\ &= \int_0^1 \nu'(\mathfrak{x})\nu'(\mathfrak{x})d\mathfrak{x} - 2\int_0^1 \mathfrak{f}(\mathfrak{x})\nu(\mathfrak{x})d\mathfrak{x} \\ &= \langle T\nu, \nu\rangle - 2\langle \mathfrak{f}, \nu\rangle, \end{aligned}$$

where

$$\langle T\nu, \nu\rangle = \int_0^1 \nu'(\mathfrak{x})\nu'(\mathfrak{x})d\mathfrak{x} \qquad (6)$$

and

$$\langle \mathfrak{f}, \nu \rangle = \int_0^1 \mathfrak{f}(\mathfrak{x})\nu(\mathfrak{x})d\mathfrak{x}. \tag{7}$$

We can finalize that the manipulator characterized through (6) is symmetric, positive, and linear. A minimal $\mathfrak{u}(\mathfrak{x})$ for operational $I[\nu]$ specified above, on the convex and closed set $\mathfrak{K}$ of $H_0^2(\Omega)$ can be described through a changing inequity of the type given below in (8) as the notion of Stampacchia and Lewy [2], equation (5) may be expressed like:

$$\left.\begin{aligned} &-\mathfrak{u}''(\mathfrak{x}) + \mu\{(\mathfrak{u}(\mathfrak{x}) - \Psi(\mathfrak{x}))\}(\mathfrak{u}(\mathfrak{x}) - \Psi(\mathfrak{x})) = f, 0 < \mathfrak{x} < 1 \\ &\mathfrak{u}(0) = 0, \mathfrak{u}'(0) = 0, \mathfrak{u}'(1) = 0 \end{aligned}\right\} \tag{8}$$

Where

$$\mu(t) = \begin{cases} -1, t \geq 0 \\ 0, t < 0 \end{cases} \tag{9}$$

represents the penalty function, and an obstacle problem $\Psi(\mathfrak{x})$ is defined as:

$$\Psi(\mathfrak{x}) = \begin{cases} -1 for 0 \leq \mathfrak{x} < \frac{1}{4} \\ 1 for \frac{1}{4} \leq \mathfrak{x} < \frac{3}{4} \\ -1 for \frac{3}{4} \leq \mathfrak{x} \leq 1 \end{cases} \tag{10}$$

From equations (8), (9), and (10), we obtained the following system of 2nd order differential equations.

$$\mathfrak{u}''(\mathfrak{x}) = \begin{cases} \mathfrak{f} for 0 \leq \mathfrak{x} < \frac{1}{4} \\ \mathfrak{u}(\mathfrak{x}) + \mathfrak{f} - 1 for \frac{1}{4} \leq x < \frac{3}{4} \\ \mathfrak{f} for \frac{3}{4} \leq x \leq 1 \end{cases} \tag{11}$$

with the BCs

$$\mathfrak{u}(0) = 0, \mathfrak{u}'(0) = 0, \mathfrak{u}'(1) = 0$$

and conditions that the function $\mathfrak{u}(\mathfrak{x})$ and $\mathfrak{u}'(\mathfrak{x})$ are continuous at $x = \frac{1}{4}$ and $x = \frac{3}{4}$.

3. Cantilinear Bridge Problem

The motivation behind studying obstacle problems originates from practical challenges in civil engineering, particularly in the modeling and analysis of large-scale bridge structures. Cantilever bridges, such as the Saint John River Bridge in New Brunswick, the Niagara River Bridge linking New York and Ontario, the Fraser River Bridge in British Columbia, and the iconic Forth Bridge in Scotland, exemplify the complexity of such structures. Similarly, the Howrah Bridge across the Hooghly River in West Bengal, India, stands as one of the largest cantilever bridges in the world, representing a remarkable achievement of structural engineering.

The formulation of these bridges reflects the ingenuity of civil engineers who have employed advanced mathematical concepts to translate physical realities into solvable mathematical models [19]. To ensure safety, durability, and sustainability, it is crucial to represent these systems in a way that allows simulations of real-world conditions, thereby mitigating unexpected risks. In this context, variational inequalities play a fundamental role, enabling engineers to design and analyze these structures as *linear obstacle problems* [1,2]. This framework not only supports the accurate modeling of load distribution and stress but also strengthens the predictive capacity of mathematical models in structural engineering.

Recent advancements have emphasized the need for innovative strategies to improve the accuracy and convergence of mathematical formulations related to cable-supported and multi-span bridges. Unlike traditional numerical techniques, modern methods—such as decomposition approaches and semi-analytic techniques—provide more precise and rapidly convergent solutions. These methods have been particularly effective in addressing multi-support systems where conventional approaches often face limitations [7]. Such creative techniques enable the reliable approximation of complex structural behaviors, ensuring that the computed solutions align closely with physical expectations.

A major challenge lies in handling variational inequalities arising in multi-support bridge models. Stampacchia's work on the regularity of solutions [2], along with Lewy's pioneering contributions, laid the foundation for treating such problems as systems of differential equations. This framework is particularly advantageous because it simplifies the computational treatment of otherwise intricate models. The incorporation of Sobolev space theory [3] and Hilbert space methods [4] has further enriched the mathematical background required to handle these problems effectively.

Over the years, several researchers have extended these ideas by developing spline-based and collocation methods. Noor and Khalifa [11-13], Al-Said and Noor [10], and Tirmizi et al. [20] introduced computational approaches ranging from cubic and quintic spline methods to finite difference schemes for solving obstacle and boundary-value problems of higher orders. Similarly, researchers like Khan and Aziz [6,12], Siddiqi and Akram [15,16], and others have advanced non-polynomial and parametric spline techniques, which have proven effective in modeling higher-order systems. These methods have demonstrated remarkable efficiency in simulating complex real-world structures, particularly those involving multi-span and cable-supported configurations.

In summary, the mathematical modeling of cantilever and multi-support bridges through obstacle problems has evolved into a rich field of study. From foundational concepts of variational inequalities [21] to advanced spline and finite element techniques [14], researchers have continuously refined the theoretical and computational tools needed to capture the intricacies of structural engineering problems. In the following section, we present exact solutions to selected obstacle problems and explore the challenges associated with obtaining such solutions.

4. Exact Solution of Obstacle Problems

In this section, we will find the different obstacle problems for different values of the continuous functions $\mathfrak{g}(\mathfrak{x})$, $\mathfrak{f}(\mathfrak{x})$, $\mathfrak{a}, \mathfrak{b}, \mathfrak{r}, \ \mathfrak{c}, \mathfrak{d}$, and the corresponding boundary conditions. We will find exact solutions to the designed Obstacle Problems. This section will also show the graph of the exact solution of the considered Obstacle Problems. This section will also discover the estimated result of the considered Obstacle equations by considering OHAM and compare them with the exact solution.

Example 4.1: Consider the following 2nd order obstacle problem

$$\mathfrak{u}^{(2)}(\mathfrak{x}) = \begin{cases} \mathfrak{g}(\mathfrak{x}), \mathfrak{d} \leq \mathfrak{x} < \mathfrak{b} \\ \mathfrak{f}(\mathfrak{x})\mathfrak{u}(\mathfrak{x}) + \mathfrak{g}(\mathfrak{x}) + \mathfrak{r}, \mathfrak{c} \leq \mathfrak{x} < \mathfrak{d} \\ \mathfrak{g}(\mathfrak{x}), \mathfrak{a} \ \leq \mathfrak{x} < \mathfrak{c} \end{cases}$$

with $\mathfrak{g}(\mathfrak{x}) = 0$, $\mathfrak{f}(\mathfrak{x}) = 1$, $\mathfrak{r} = -1$, $\mathfrak{a} = -1, \mathfrak{c} = -\frac{1}{2}, \mathfrak{d} = \frac{1}{2}$ and $\mathfrak{b} = 1$. The BCs given are $(1) = \mathfrak{u}(-1) = 0$.

To find the precise result for the above example, we can proceed in the following way:

First, we consider the differential equations.

$\mathfrak{u}_0{}^{(2)}(\mathfrak{x}) = 0, \mathfrak{u}_0(-1) = 0$ and solve with respect to $\mathfrak{x}$ for $\mathfrak{u}_0(\mathfrak{x})$

$\mathfrak{u}_1{}^{(2)}(\mathfrak{x}) = \mathfrak{u}_1(\mathfrak{x}) - 1$and solve with respect to $\mathfrak{x}$ for $\mathfrak{u}_1(\mathfrak{x})$

$\mathfrak{u}_2{}^{(2)}(\mathfrak{x}) = 0, \mathfrak{u}_2(1) = 0$and solve with respect to $\mathfrak{x}$ for $\mathfrak{u}_2(\mathfrak{x})$

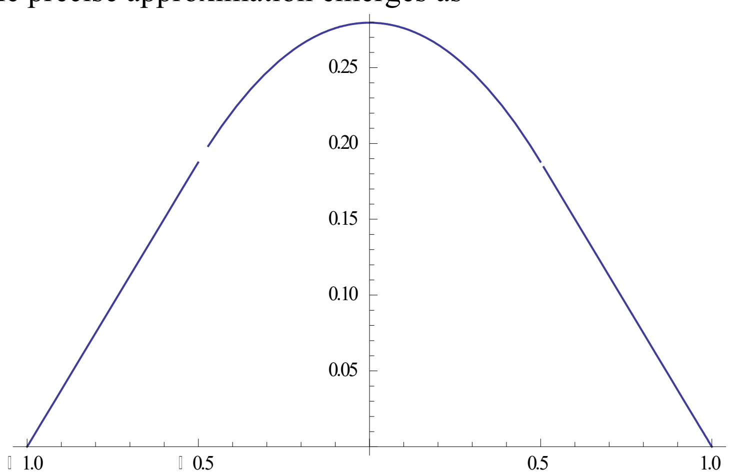
By solving each differential equation by the method of determining coefficients, we will get
$\mathfrak{u}_0(\mathfrak{x}) = (1+\mathfrak{x})a_1$
$\mathfrak{u}_1(\mathfrak{x}) = 1 + a_2 e^{\mathfrak{x}} + a_3 e^{-\mathfrak{x}}$
$\mathfrak{u}_2(\mathfrak{x}) = a_4(1-\mathfrak{x})$
In this system of solutions, we have to find the constants a_1, a_2, a_3 and a_4 using the condition of the functions $\mathbf{u}^{(1)}(\mathbf{x})$ is continuous on $\mathfrak{c}$ and $\mathfrak{d}$. To compute the values of a_1, a_2, a_3 and a_4, the Gauss elimination method is used as:

$$\mathfrak{u}_0'(-\frac{1}{2}) - \mathfrak{u}_1'(-\frac{1}{2}) = 0$$

$\mathfrak{u}_1'(\frac{1}{2}) - \mathfrak{u}_2'(\frac{1}{2}) = 0$
After finding the values, the accurate results are as follows:

$$\mathfrak{u}(\mathfrak{x}) = \begin{cases} \dfrac{2(-1+e)(1+\mathfrak{x})}{1+3e}, \mathfrak{a} \le \mathfrak{x} < \mathfrak{c} \\ 1 - \dfrac{4\sqrt{e}\,cosh[\mathfrak{x}]}{1+3e}, \mathfrak{c} \le \mathfrak{x} < \mathfrak{d} \\ -\dfrac{2(-1+e)(-1+\mathfrak{x})}{1+3e}, \mathfrak{d} \le \mathfrak{x} < \mathfrak{b} \end{cases}$$

The graphical view of the precise approximation emerges as



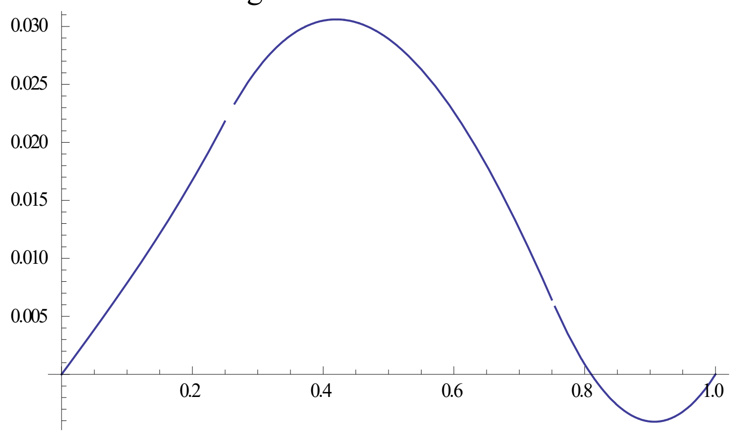
Example 4.2: Consider the following 2nd-order obstacle problem

$$\mathfrak{u}^{(2)}(\mathfrak{x}) = \begin{cases} \mathfrak{g}(\mathfrak{x}), \mathfrak{a} \le \mathfrak{x} < \mathfrak{c} \\ \mathfrak{f}(\mathfrak{x})\mathfrak{u}(\mathfrak{x}) + \mathfrak{g}(\mathfrak{x}) + \mathfrak{r}, \mathfrak{c} \le \mathfrak{x} < \mathfrak{d} \\ \mathfrak{g}(\mathfrak{x}), \mathfrak{d} \le \mathfrak{x} < \mathfrak{b} \end{cases}$$

with $\mathfrak{g}(\mathfrak{x}) = \mathfrak{x}$, $\mathfrak{f}(\mathfrak{x}) = 1$, $\mathfrak{r} = -1$, $\mathfrak{a} = 0, \mathfrak{c} = \frac{1}{4}, \mathfrak{d} = \frac{3}{4}$ and $\mathfrak{b} = 1$. The BCs given are $\mathfrak{u}(0) = \mathfrak{u}(1) = 0$. Working as in the previous example, the exact solution is given as

$$\mathfrak{u}(\mathfrak{x}) = \begin{cases} -\dfrac{\left(-2049+80\sqrt{e}+545e\right)x}{96(-9+25e)} + \dfrac{x^3}{6}, \mathfrak{a} \leq \mathfrak{x} < \mathfrak{c} \\ 1 + \dfrac{e^{-\frac{1}{4}-x}\left(15e - 965e^{3/2} + 579e^{2x} - 25e^{\frac{1}{2}+2x}\right)}{48(-9+25e)} - x, \mathfrak{c} \leq \mathfrak{x} < \mathfrak{d} \\ \dfrac{1}{6}(-1+x)\left(\dfrac{933+3088\sqrt{e}-2725e}{-144+400e} + x + x^2\right), \mathfrak{d} \leq \mathfrak{x} < \mathfrak{b} \end{cases}$$

The graphical view of the exact solution is given as



Example 4.3: Consider the following 2nd-order obstacle problem

$$\mathfrak{u}^{(2)}(\mathfrak{x}) = \begin{cases} \mathfrak{g}(\mathfrak{x}), \mathfrak{a} \leq \mathfrak{x} < \mathfrak{c} \\ \mathfrak{f}(\mathfrak{x})\mathfrak{u}(\mathfrak{x}) + \mathfrak{g}(\mathfrak{x}) + \mathfrak{r}, \mathfrak{c} \leq \mathfrak{x} < \mathfrak{d} \\ \mathfrak{g}(\mathfrak{x}), \mathfrak{d} \leq \mathfrak{x} < \mathfrak{b} \end{cases}$$

with $\mathfrak{g}(\mathfrak{x}) = u'(\mathfrak{x}) - 2$, $\mathfrak{f}(\mathfrak{x}) = 1$, $\mathfrak{r} = -1$, $\mathfrak{a} = 0$, $\mathfrak{c} = \frac{1}{4}$, $\mathfrak{d} = \frac{3}{4}$ and $\mathfrak{b} = 1$. And boundary constraints given are $\mathfrak{u}(1) = \mathfrak{u}(0) = 0$.

Working as in the previous example, accurate results are as follows:

$$\mathfrak{u}(\mathfrak{x}) = \begin{cases} 2x + (-1+e^x)a_1, \mathfrak{a} \leq \mathfrak{x} < \mathfrak{c} \\ 3 + e^{-\frac{1}{2}(-1+\sqrt{5})x}\left(f + ge^{\sqrt{5}x}\right), \mathfrak{c} \leq \mathfrak{x} < \mathfrak{d} \\ -2 + 2x + (-e+e^x)a_2, \mathfrak{d} \leq \mathfrak{x} < \mathfrak{b} \end{cases}$$

where

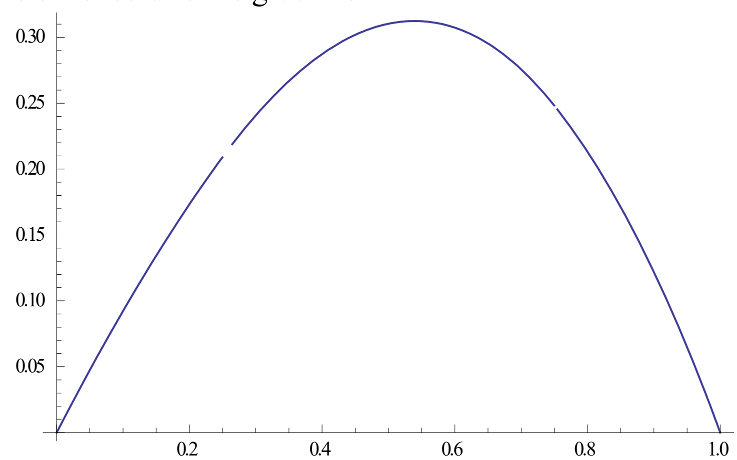
$$a_1 = \frac{-11\sqrt{5} + e^{1/4}\left(4\sqrt{5} + \sqrt{5}\left(9 - 4e^{1/4}\right) cosh\left[\frac{\sqrt{5}}{4}\right] + \left(-19 + 6e^{1/4}\right) sinh\left[\frac{\sqrt{5}}{4}\right]\right)}{2e^{1/4}\left(\sqrt{5}\left(-1+\sqrt{e}\right) cosh\left[\frac{\sqrt{5}}{4}\right] + \left(3 - 4e^{1/4} + 3\sqrt{e}\right) sinh\left[\frac{\sqrt{5}}{4}\right]\right)}$$

$$f = \frac{\left(e^{-\frac{5}{8}+\frac{3\sqrt{5}}{8}}\left(-\left(-11+4e^{1/4}\right)\left(5+\sqrt{5}+\left(-5+\sqrt{5}\right)e^{1/4}\right)e^{1/4} - \left(-4+9e^{1/4}\right)\left(-5+\sqrt{5}+\left(5+\sqrt{5}\right)e^{1/4}\right)e^{\frac{1}{4}(2+\sqrt{5})}\right)\right)}{\left(2\sqrt{5}\left(-3-\sqrt{5}+4e^{1/4}-3\sqrt{e}+\left(3-\sqrt{5}-4e^{1/4}+\left(3+\sqrt{5}\right)\sqrt{e}\right)e^{\frac{\sqrt{5}}{2}}+\sqrt{5}e\right)\right)}$$

$$g=\frac{\left(e^{\frac{1}{8}(-5-\sqrt{5})}\begin{pmatrix}-4(1+\sqrt{5})\sqrt{e}+(5+13\sqrt{5})e^{\frac{3}{4}}-9(-1+\sqrt{5})e+\\ \left(-11+4e^{\frac{1}{4}}\right)\left(1-\sqrt{5}+(1+\sqrt{5})e^{\frac{1}{4}}\right)e^{\frac{1}{4}(1+\sqrt{5})}\end{pmatrix}\right)}{\left(2\left(-3-\sqrt{5}+4e^{\frac{1}{4}}-3\sqrt{e}+\left(3-\sqrt{5}-4e^{\frac{1}{4}}+(3+\sqrt{5})\sqrt{e}\right)e^{\frac{\sqrt{5}}{2}}+\sqrt{5e}\right)\right)}$$

$$a_2=\frac{\left(-10+4\sqrt{5}+(25-11\sqrt{5})e^{1/4}+(10+4\sqrt{5}-(25+11\sqrt{5})e^{1/4})e^{\frac{\sqrt{5}}{2}}+2\sqrt{5}(-4+9e^{1/4})e^{\frac{1}{4}(1+\sqrt{5})}\right)}{\left(-2(3+\sqrt{5})e^{3/4}+8e+2(-3+\sqrt{5})e^{5/4}+2(3-\sqrt{5}-4e^{1/4}+(3+\sqrt{5})\sqrt{e})e^{\frac{3}{4}+\frac{\sqrt{5}}{2}}\right)}$$

The graphical view of the exact solution is given as



Example 4.4: Consider the following 2nd-order obstacle problem

$$\mathfrak{u}^{(2)}(\mathfrak{x})=\begin{cases}\mathfrak{g}(\mathfrak{x}), \mathfrak{d}\le\mathfrak{x}<\mathfrak{b}\\ \mathfrak{f}(\mathfrak{x})\mathfrak{u}(\mathfrak{x})+\mathfrak{g}(\mathfrak{x})+\mathfrak{r}, \mathfrak{c}\le\mathfrak{x}<\mathfrak{d}\\ \mathfrak{g}(\mathfrak{x}), \mathfrak{a}\le\mathfrak{x}<c\end{cases}$$

with $\mathfrak{g}(\mathfrak{x})=0$, $\mathfrak{f}(\mathfrak{x})=1$, $\mathfrak{r}=-1$, $\mathfrak{a}=0, \mathfrak{c}=\frac{\pi}{4}, \mathfrak{d}=\frac{3\pi}{4}$ and $\mathfrak{b}=\pi$. The BCs given are $\mathfrak{u}(0)=\mathfrak{u}(1)=0$. Working as in the previous, the exact solution is:

$$\mathfrak{u}(\mathfrak{x})=\begin{cases}a_1x, \mathfrak{a}\le\mathfrak{x}<\mathfrak{c}\\ 1+e^{x}a_2+e^{-x}a_3, \mathfrak{c}\le\mathfrak{x}<\mathfrak{d}\\ \dfrac{a_4(\pi-x)}{\pi}, \mathfrak{d}\le\mathfrak{x}<\mathfrak{b}\end{cases}$$

Where

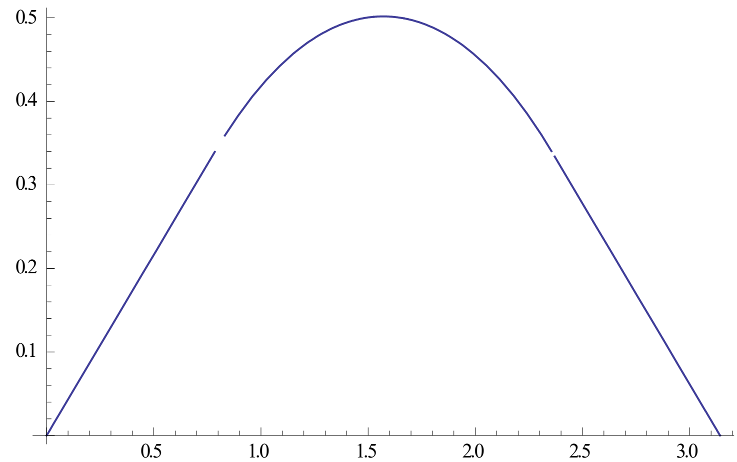
$$a_1=\frac{4}{\pi+4\,coth\left[\frac{\pi}{4}\right]}$$

$$a_2=-\frac{4e^{-\pi/4}}{4-\pi+e^{\pi/2}(4+\pi)}$$

$$a_3=-\frac{4e^{3\pi/4}}{4-\pi+e^{\pi/2}(4+\pi)}$$

$$a_4=\frac{4\pi}{\pi+4\,coth\left[\frac{\pi}{4}\right]}$$

The graphical view of the accurate results is as follows:



Example 4.5: Suppose the following 2^{nd}-order obstacle problem

$$-\mathfrak{u}^{(2)}(\mathfrak{x}) = \begin{cases} \mathfrak{g}(\mathfrak{x}), \mathfrak{d} \leq \mathfrak{x} < \mathfrak{b} \\ \mathfrak{f}(\mathfrak{x})\mathfrak{u}(\mathfrak{x}) + \mathfrak{g}(\mathfrak{x}) + \mathfrak{r}, \mathfrak{c} \leq \mathfrak{x} < \mathfrak{d} \\ \mathfrak{g}(\mathfrak{x}), \mathfrak{a} \leq \mathfrak{x} < \mathfrak{c} \end{cases}$$

with $\mathfrak{g}(\mathfrak{x}) = \mathfrak{u} + 1$, $\mathfrak{f}(\mathfrak{x}) = 1$, $\mathfrak{r} = -1$, $\mathfrak{a} = 0, \mathfrak{c} = \frac{\pi}{4}, \mathfrak{d} = \frac{3\pi}{4}$ and $\mathfrak{b} = \pi$. The boundary-terms given are $\mathfrak{u}(0) = \mathfrak{u}(1) = 0$.

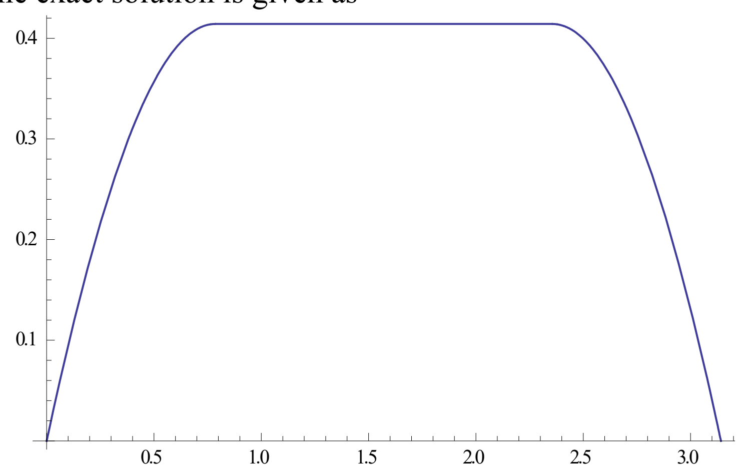
Working as in the previous example, the exact solution is given as

$$\mathfrak{u}(\mathfrak{x}) = \begin{cases} -1 + cos[\mathfrak{x}] + \left(-1 + \sqrt{2} + \sqrt{2}k\right) sin[\mathfrak{x}], \mathfrak{a} \leq \mathfrak{x} < \mathfrak{c} \\ \dfrac{3k\pi - p\pi - 4k\mathfrak{x} + 4p\mathfrak{x}}{2\pi}, \mathfrak{c} \leq \mathfrak{x} < \mathfrak{d} \\ -1 - cos[\mathfrak{x}] + \left(-1 + \sqrt{2} + \sqrt{2}p\right) sin[\mathfrak{x}], \mathfrak{d} \leq \mathfrak{x} < \mathfrak{b} \end{cases}$$

Where

$$k = -1 + \sqrt{2}$$
$$p = -1 + \sqrt{2}$$

The graphical view of the exact solution is given as



Example 4.6: Assume the following 3^{rd}-order obstacle

$$\mathfrak{u}^{(3)}(\mathfrak{x}) = \begin{cases} \mathfrak{g}(\mathfrak{x}), \mathfrak{d} \leq \mathfrak{x} < \mathfrak{b} \\ \mathfrak{f}(\mathfrak{x})\mathfrak{u}(\mathfrak{x}) + \mathfrak{g}(\mathfrak{x}) + \mathfrak{r}, \mathfrak{c} \leq \mathfrak{x} < \mathfrak{d} \\ \mathfrak{g}(\mathfrak{x}), \mathfrak{a} \leq \mathfrak{x} < \mathfrak{c} \end{cases}$$

with $\mathfrak{g}(\mathfrak{x}) = 0$, $\mathfrak{f}(\mathfrak{x}) = 1$, $\mathfrak{r} = -1$, $\mathfrak{a} = 0, \mathfrak{c} = \frac{1}{4}, \mathfrak{d} = \frac{3}{4}$ and $\mathfrak{b} = 1$. And boundary constraints given are $\mathfrak{u}(1) = \mathfrak{u}(0) = 0$, $\mathfrak{u}'(\frac{3}{4}) = \mathfrak{u}'(\frac{1}{4}) = 0$.

To compute the exact solution to problem 5, we can proceed in the following way:

First, we consider the differential equations.

${\mathfrak{u}_0}^{(3)}(\mathfrak{x}) = 0, \mathfrak{u}_0(1) = 0$ and solve with respect to $\mathfrak{x}$ for $\mathfrak{u}_0(\mathfrak{x})$

${\mathfrak{u}_1}^{(3)}(\mathfrak{x}) = \mathfrak{u}_1(\mathfrak{x}) - 1$and solve with respect to $\mathfrak{x}$ for $\mathfrak{u}_1(\mathfrak{x})$

${\mathfrak{u}_2}^{(3)}(\mathfrak{x}) = 0, \mathfrak{u}_2(1) = 0$ and solve with respect to $\mathfrak{x}$ for $\mathfrak{u}_2(\mathfrak{x})$

By solving each differential equation by the method of undetermined coefficients, we will get

$\mathfrak{u}_0(\mathfrak{x}) = \mathfrak{x}(a + b\mathfrak{x})$

$\mathfrak{u}_1(\mathfrak{x}) = 1 + e^{\mathfrak{x}}t + e^{-\mathfrak{x}/2}\left(p \cos\left[\frac{\sqrt{3}\mathfrak{x}}{2}\right] + q \sin\left[\frac{\sqrt{3}\mathfrak{x}}{2}\right]\right)$

$\mathfrak{u}_2(\mathfrak{x}) = (1 - \mathfrak{x})\big(d\mathfrak{x} + c(1 + \mathfrak{x})\big)$

In this system of solutions, we have to find the constants a, b, c, d, t, q, p using the condition of the functions $\mathbf{u}^{(1)}(\mathbf{x})$ is continuous on $\mathfrak{c}$ and $\mathfrak{d}$. To calculate a, b, c, d, t, q, and p, Gauss elimination method is used as:

$$\mathfrak{u}_0'[1/4] - \mathfrak{u}_1'[1/4] == 0, \mathfrak{u}_1'[3/4] - \mathfrak{u}_2'[3/4] == 0$$

$\mathfrak{u}_0''[1/4] - \mathfrak{u}_1''[1/4] == 0, \mathfrak{u}_1''[3/4] - \mathfrak{u}_2''[3/4] == 0$

After finding the values, the accurate results are as follows:

$$\mathfrak{u}(\mathfrak{x}) = \begin{cases} \mathfrak{x}(a + b\mathfrak{x}), \mathfrak{a} \leq \mathfrak{x} < \mathfrak{c} \\ 1 + e^{\mathfrak{x}}t + e^{-\mathfrak{x}/2}\left(p \cos\left[\frac{\sqrt{3}\mathfrak{x}}{2}\right] + q \sin\left[\frac{\sqrt{3}\mathfrak{x}}{2}\right]\right), \mathfrak{c} \leq \mathfrak{x} < \mathfrak{d} \\ (1 - \mathfrak{x})\big(d\mathfrak{x} + c(1 + \mathfrak{x})\big), \mathfrak{d} \leq \mathfrak{x} < \mathfrak{b} \end{cases}$$

Where, if $c = 1$ we can have

$$a = \frac{\left(\begin{array}{l} 888 + 41(24 - 41c)e + \left(-984 + 1297c - 888\sqrt{e}\right)e^{1/4} \cos\left[\frac{\sqrt{3}}{4}\right] + \\ \sqrt{3}\left(1048 + 527c - 1208\sqrt{e}\right)e^{1/4} \sin\left[\frac{\sqrt{3}}{4}\right] \end{array}\right)}{\left(-925 + e^{3/4}\left(1309 \cos\left[\frac{\sqrt{3}}{4}\right] - 1059\sqrt{3} \sin\left[\frac{\sqrt{3}}{4}\right]\right)\right)}$$

$$b = \frac{\left(4\left(148 + 5(-24 + 41c)e + \left(-264 - 13c + 236\sqrt{e}\right)e^{\frac{1}{4}} \cos\left[\frac{\sqrt{3}}{4}\right] + \sqrt{3}\left(8 + 117c + 44\sqrt{e}\right)e^{\frac{1}{4}} \sin\left[\frac{\sqrt{3}}{4}\right]\right)\right)}{\left(-925 + e^{\frac{3}{4}}\left(1309 \cos\left[\frac{\sqrt{3}}{4}\right] - 1059\sqrt{3} \sin\left[\frac{\sqrt{3}}{4}\right]\right)\right)}$$

$$q = \frac{\left(e^{3/8}(B_1 - B_2 + B_3)\right)}{\left(-2775 + e^{3/4}\left(3927 \cos\left[\frac{\sqrt{3}}{4}\right] - 3177\sqrt{3} \sin\left[\frac{\sqrt{3}}{4}\right]\right)\right)}$$

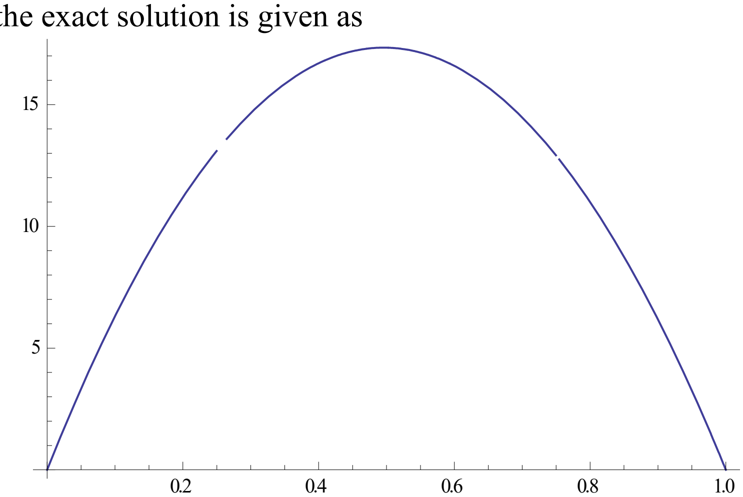
where

$$B_1 = \sqrt{3}\left(600 + 1375c - 2304\sqrt{e}\right) cos\left[\frac{3\sqrt{3}}{8}\right]$$
$$B_2 = (-24 + 41c)e^{3/4}\left(71\sqrt{3}\, cos\left[\frac{\sqrt{3}}{8}\right] + 27\, sin\left[\frac{\sqrt{3}}{8}\right]\right)$$
$$B_3 = 75(40 - 7c)\, sin\left[\frac{3\sqrt{3}}{8}\right] - 4416\sqrt{e}\, sin\left[\frac{3\sqrt{3}}{8}\right]$$
$$d = \frac{400 + 1225c - 1536\sqrt{e} + (1136 - 2377c)e^{\frac{3}{4}} cos\left[\frac{\sqrt{3}}{4}\right] + \sqrt{3}(-144 + 599c)e^{\frac{3}{4}} sin\left[\frac{\sqrt{3}}{4}\right]}{-925 + e^{\frac{3}{4}}\left(1309\, cos\left[\frac{\sqrt{3}}{4}\right] - 1059\sqrt{3}\, sin\left[\frac{\sqrt{3}}{4}\right]\right)}$$
$$t = \frac{3552 + 2e^{1/4}\left(24(-82 + 31c)\, cos\left[\frac{\sqrt{3}}{4}\right] + \sqrt{3}(696 + 929c)\, sin\left[\frac{\sqrt{3}}{4}\right]\right)}{-2775e^{1/4} + 3927e\, cos\left[\frac{\sqrt{3}}{4}\right] - 3177\sqrt{3}e\, sin\left[\frac{\sqrt{3}}{4}\right]}$$
$$p = \frac{\left(e^{3/8}(A1 + A2 + A3)\right)}{\left(-2775 + e^{3/4}\left(3927\, cos\left[\frac{\sqrt{3}}{4}\right] - 3177\sqrt{3}\, sin\left[\frac{\sqrt{3}}{4}\right]\right)\right)}$$

where

$$A1 = 75(40 - 7c)\, cos\left[\frac{3\sqrt{3}}{8}\right] - 4416\sqrt{e}\, cos\left[\frac{3\sqrt{3}}{8}\right]$$
$$A2 = (-24 + 41c)e^{3/4}\left(-27\, cos\left[\frac{\sqrt{3}}{8}\right] + 71\sqrt{3}\, sin\left[\frac{\sqrt{3}}{8}\right]\right)$$
$$A3 = \sqrt{3}\left(-600 - 1375c + 2304\sqrt{e}\right) sin\left[\frac{3\sqrt{3}}{8}\right]$$

The graphical view of the exact solution is given as



Example 4.7: Suppose the following 3rd order obstacle problem

$$\mathfrak{u}^{(3)}(\mathfrak{x}) = \begin{cases} \mathfrak{g}(\mathfrak{x}), \mathfrak{d} \leq \mathfrak{x} < \mathfrak{b} \\ \mathfrak{f}(\mathfrak{x})\mathfrak{u}(\mathfrak{x}) + \mathfrak{g}(\mathfrak{x}) + \mathfrak{r}, \mathfrak{c} \leq \mathfrak{x} < \mathfrak{d} \\ \mathfrak{g}(\mathfrak{x}), \mathfrak{a} \leq \mathfrak{x} < \mathfrak{c} \end{cases}$$

with $\mathfrak{g}(\mathfrak{x}) = 2$, $\mathfrak{f}(\mathfrak{x}) = 1$, $\mathfrak{r} = 1$, $\mathfrak{a} = 0, \mathfrak{c} = \frac{1}{4}, \mathfrak{d} = \frac{3}{4}$ and $\mathfrak{b} = 1$. The BCs given are $\mathfrak{u}(0) = \mathfrak{u}(1) = 0$, $\mathfrak{u}'(\frac{1}{4}) = \mathfrak{u}'(\frac{3}{4}) = 0$.

After finding the values, the exact solution is given as

$$\mathfrak{u}(\mathfrak{x}) = \begin{cases} \frac{1}{3}\mathfrak{x}(3a + 3b\mathfrak{x} + \mathfrak{x}^2), \mathfrak{a} \le \mathfrak{x} < \mathfrak{c} \\ 1 + e^{\mathfrak{x}}t + e^{-\mathfrak{x}/2}\left(p\, cos\left[\frac{\sqrt{3}\mathfrak{x}}{2}\right] + q\, sin\left[\frac{\sqrt{3}\mathfrak{x}}{2}\right]\right), \mathfrak{c} \le \mathfrak{x} < \mathfrak{d} \\ \frac{1}{3}(-1 + \mathfrak{x})(-3c - 3(c + \mathfrak{d})\mathfrak{x} + \mathfrak{x}^2), \mathfrak{d} \le \mathfrak{x} < \mathfrak{b} \end{cases}$$

Where, if $c = 1$ we can have

$$a = \frac{-(39627 + X_1 + X_2 + X_3)}{\left(48\left(925 + e^{3/4}\left(-1309\, cos\left[\frac{\sqrt{3}}{4}\right] + 1059\sqrt{3}\, sin\left[\frac{\sqrt{3}}{4}\right]\right)\right)\right)}$$

Where

$$X_1 = 41(871 - 1968c)e$$

$$X_2 = \left(-38399 + 62256c - 38475\sqrt{e}\right)e^{1/4}\, cos\left[\frac{\sqrt{3}}{4}\right]$$

$$X_3 = \sqrt{3}\left(54255 + 25296c - 60859\sqrt{e}\right)e^{1/4}\, sin\left[\frac{\sqrt{3}}{4}\right]$$

$$b = \frac{-(9842 - Y_1 + Y_2 + Y_3)}{\left(12\left(925 + e^{3/4}\left(-1309\, cos\left[\frac{\sqrt{3}}{4}\right] + 1059\sqrt{3}\, sin\left[\frac{\sqrt{3}}{4}\right]\right)\right)\right)}$$

Where

$$Y_1 = (4355 + 9840c)e$$

$$Y_2 = \left(-12829 - 624c + 7342\sqrt{e}\right)e^{1/4}\, cos\left[\frac{\sqrt{3}}{4}\right]$$

$$Y_3 = \sqrt{3}\left(1205 + 5616c + 5278\sqrt{e}\right)e^{1/4}\, sin\left[\frac{\sqrt{3}}{4}\right]$$

$$t = \frac{84804 + e^{1/4}\left(36(-2493 + 992c)\, cos\left[\frac{\sqrt{3}}{4}\right] + \sqrt{3}(40085 + 44592c)\, sin\left[\frac{\sqrt{3}}{4}\right]\right)}{72\left(-925e^{1/4} + 1309e\, cos\left[\frac{\sqrt{3}}{4}\right] - 1059\sqrt{3}e\, sin\left[\frac{\sqrt{3}}{4}\right]\right)}$$

$$p = \frac{\left(e^{3/8}(W_1 + W_2 + W_3)\right)}{\left(144\left(925 + e^{3/4}\left(-1309\, cos\left[\frac{\sqrt{3}}{4}\right] + 1059\sqrt{3}\, sin\left[\frac{\sqrt{3}}{4}\right]\right)\right)\right)}$$

Where

$$W_1 = 225(627 - 112c)\cos\left[\frac{3\sqrt{3}}{8}\right]$$

$$W_2 = -210864\sqrt{e}\cos\left[\frac{3\sqrt{3}}{8}\right]$$

$$W_3 = (-871 + 1968c)e^{3/4}(-27\cos\left[\frac{\sqrt{3}}{8}\right] + 71\sqrt{3}\sin\left[\frac{\sqrt{3}}{8}\right]$$

$$q = \frac{\left(e^{3/8}(Z_1 + Z_2 + Z_3 + Z_4)\right)}{\left(48\left(-925\sqrt{3} + e^{3/4}\left(1309\sqrt{3}\cos\left[\frac{\sqrt{3}}{4}\right] - 3177\sin\left[\frac{\sqrt{3}}{4}\right]\right)\right)\right)}$$

$$Z_1 = 25(1543 + 2640c)\cos\left[\frac{3\sqrt{3}}{8}\right]$$

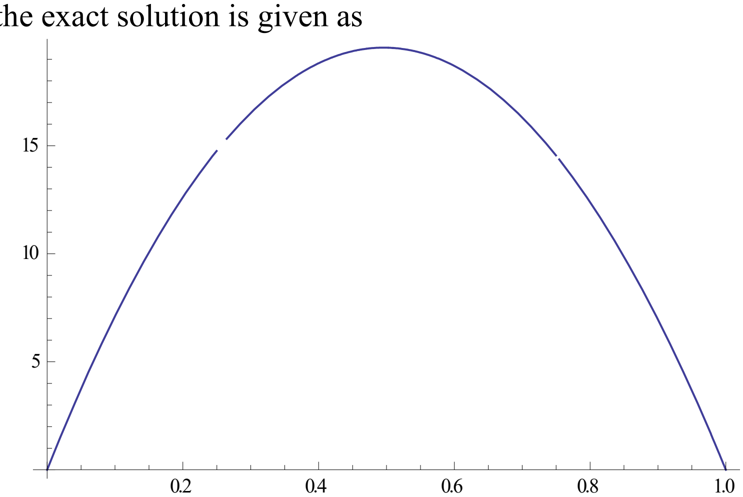
$$Z_2 = -110016\sqrt{e}\cos\left[\frac{3\sqrt{3}}{8}\right]$$

$$Z_3 = (-871 + 1968c)e^{3/4}\left(71\cos\left[\frac{\sqrt{3}}{8}\right] + 9\sqrt{3}\sin\left[\frac{\sqrt{3}}{8}\right]\right)$$

$$Z_4 = \left[\sqrt{3}\left(47025 - 8400c - 70288\sqrt{e}\right)\sin\left[\frac{3\sqrt{3}}{8}\right]\right]$$

$$d = \frac{\left(\begin{array}{l}\sqrt{3}\left(725 + 14700c - 18336\sqrt{e}\right) + \sqrt{3}(18379 - 28524c)e^{3/4}\cos\left[\frac{\sqrt{3}}{4}\right] + \\ 3(-7837 + 7188c)e^{3/4}\sin\left[\frac{\sqrt{3}}{4}\right]\end{array}\right)}{\left(12\left(-925\sqrt{3} + e^{3/4}\left(1309\sqrt{3}\cos\left[\frac{\sqrt{3}}{4}\right] - 3177\sin\left[\frac{\sqrt{3}}{4}\right]\right)\right)\right)}$$

The graphical view of the exact solution is given as



Example 4.8: Assume the following 3rd order obstacle problem

$$\mathfrak{u}^{(3)}(\mathfrak{x}) = \begin{cases} \mathfrak{g}(\mathfrak{x}), \mathfrak{a} \leq \mathfrak{x} < \mathfrak{c} \\ \mathfrak{f}(\mathfrak{x})\mathfrak{u}(\mathfrak{x}) + \mathfrak{g}(\mathfrak{x}) + \mathfrak{r}, \mathfrak{c} \leq \mathfrak{x} < \mathfrak{d} \\ \mathfrak{g}(\mathfrak{x}), \mathfrak{d} \leq \mathfrak{x} < \mathfrak{b} \end{cases}$$

With $\mathfrak{g}(\mathfrak{x}) = \mathfrak{x}$, $\mathfrak{f}(\mathfrak{x}) = 1$, $\mathfrak{r} = -1$, $\mathfrak{a} = 0, \mathfrak{c} = \frac{1}{4}, \mathfrak{d} = \frac{3}{4}$ and $\mathfrak{b} = 1$. And boundary constraints given are $\mathfrak{u}(1) = \mathfrak{u}(0) = 0, \mathfrak{u}'(\frac{3}{4}) = \mathfrak{u}'(\frac{1}{4}) = 0$.

After finding the values, the exact solution is given as

$$\mathfrak{u}(\mathfrak{x}) = \begin{cases} a_1\mathfrak{x} + a_2\mathfrak{x}^2 + \dfrac{\mathfrak{x}^4}{24}, \mathfrak{a} \leq \mathfrak{x} < \mathfrak{c} \\ 1 + e^{\mathfrak{x}}a_3 - \mathfrak{x} + e^{-\mathfrak{x}/2}\left(a_4 \cos\left[\dfrac{\sqrt{3}\mathfrak{x}}{2}\right] + a_5 \sin\left[\dfrac{\sqrt{3}\mathfrak{x}}{2}\right]\right), \mathfrak{c} \leq \mathfrak{x} < \mathfrak{d} \\ \dfrac{1}{24}(-1 + \mathfrak{x})\big(-24a_6\mathfrak{x} + (1 + \mathfrak{x})(-24a_7 + \mathfrak{x}^2)\big), \mathfrak{d} \leq \mathfrak{x} < \mathfrak{b} \end{cases}$$

If $a_7 = 1$ we can have

$$a_1 = \frac{-(1388351 + L_1 + L_2 - L_3)}{\left(768\left(925 + e^{3/4}\left(-1309 \cos\left[\frac{\sqrt{3}}{4}\right] + 1059\sqrt{3} \sin\left[\frac{\sqrt{3}}{4}\right]\right)\right)\right)}$$

Where

$$L_1 = 41(30215 - 31488c)e$$

$$L_2 = \left(-956191 + 996096c - 1681727\sqrt{e}\right)e^{1/4} \cos\left[\frac{\sqrt{3}}{4}\right]$$

$$L_3 = \left[3\sqrt{3}\left(-134912c + 2627\left(49 + 15\sqrt{e}\right)\right) e^{1/4} \sin\left[\frac{\sqrt{3}}{4}\right]\right]$$

$$a_2 = \frac{-(232767 + M_1 + M_2 + M_3)}{\left(384\left(925 + e^{3/4}\left(-1309 \cos\left[\frac{\sqrt{3}}{4}\right] + 1059\sqrt{3} \sin\left[\frac{\sqrt{3}}{4}\right]\right)\right)\right)}$$

Where

$$M_1 = (-302150 + 314880c)e$$

$$M_2 = \left(18278 - 19968c + 354465\sqrt{e}\right)e^{1/4} \cos\left[\frac{\sqrt{3}}{4}\right]$$

$$M_3 = 13\sqrt{3}\left(-13246 + 13824c + 5685\sqrt{e}\right)e^{1/4} \sin\left[\frac{\sqrt{3}}{4}\right]$$

$$a_3 = \frac{1363302 + e^{1/4}\left(18(-30599 + 31744c) \cos\left[\frac{\sqrt{3}}{4}\right] + \sqrt{3}(-682613 + 713472c) \sin\left[\frac{\sqrt{3}}{4}\right]\right)}{1152\left(-925e^{1/4} + 1309e \cos\left[\frac{\sqrt{3}}{4}\right] - 1059\sqrt{3}e \sin\left[\frac{\sqrt{3}}{4}\right]\right)}$$

$$a_4 = \frac{-\left(e^{3/8}(N_1 + N_2 + N_3 + N_4)\right)}{\left(2304\left(925 + e^{3/4}\left(-1309\cos\left[\frac{\sqrt{3}}{4}\right] + 1059\sqrt{3}\sin\left[\frac{\sqrt{3}}{4}\right]\right)\right)\right)}$$

Where

$$N_1 = 225(1739 - 1792c)\cos\left[\frac{3\sqrt{3}}{8}\right]$$

$$N_2 = -3389832\sqrt{e}\cos\left[\frac{3\sqrt{3}}{8}\right]$$

$$N_3 = (-30215 + 31488c)e^{3/4}\left(-27\cos\left[\frac{\sqrt{3}}{8}\right] + 71\sqrt{3}\sin\left[\frac{\sqrt{3}}{8}\right]\right)$$

$$N_4 = \sqrt{3}(1011025 - 1056000c + 1768608\sqrt{e})\sin\left[\frac{3\sqrt{3}}{8}\right]$$

$$a_5 = \frac{\left(e^{3/8}(O_1 + O_2 + O_3 + O_4)\right)}{\left(768\left(-925\sqrt{3} + e^{3/4}\left(1309\sqrt{3}\cos\left[\frac{\sqrt{3}}{4}\right] - 3177\sin\left[\frac{\sqrt{3}}{4}\right]\right)\right)\right)}$$

Where

$$O_1 = 25(-40441 + 42240c)\cos\left[\frac{3\sqrt{3}}{8}\right]$$

$$O_2 = -1768608\sqrt{e}\cos\left[\frac{3\sqrt{3}}{8}\right]$$

$$O_3 = -(-30215 + 31488c)e^{3/4}\left(71\cos\left[\frac{\sqrt{3}}{8}\right] + 9\sqrt{3}\sin\left[\frac{\sqrt{3}}{8}\right]\right)$$

$$O_4 = -3\sqrt{3}(-43475 + 44800c + 376648\sqrt{e})\sin\left[\frac{3\sqrt{3}}{8}\right]$$

$$a_6 = \frac{D_1 + D_2 + D_3}{\left(384\left(-925\sqrt{3} + e^{3/4}\left(1309\sqrt{3}\cos\left[\frac{\sqrt{3}}{4}\right] - 3177\sin\left[\frac{\sqrt{3}}{4}\right]\right)\right)\right)}$$

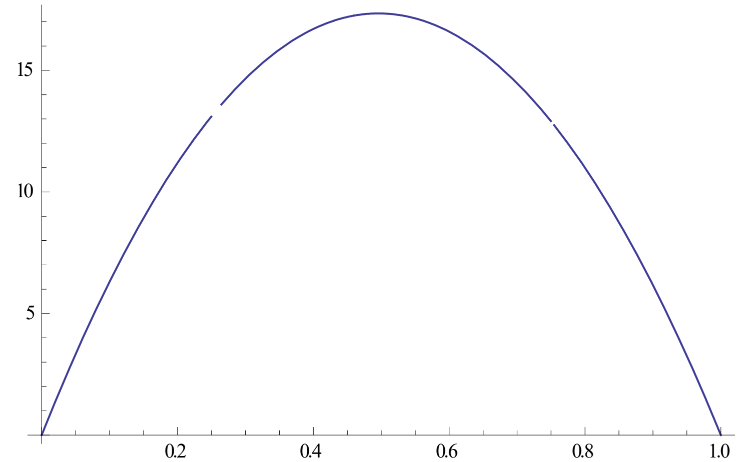
Where

$$D_1 = \sqrt{3}(-145225 + 470400c - 589536\sqrt{e})$$

$$D_2 = \sqrt{3}(443689 - 912768c)e^{3/4}\cos\left[\frac{\sqrt{3}}{4}\right]$$

$$D_3 = 3(128921 + 230016c)e^{3/4}\sin\left[\frac{\sqrt{3}}{4}\right]$$

The graphical view of the exact solution is given as



It is not always easy to find the exact solution of every choice of functions $\mathfrak{f}(\mathfrak{x})$, $\mathfrak{g}(\mathfrak{x})$, constant value $\mathfrak{r}$, $\mathfrak{a}, \mathfrak{c}, \mathfrak{d}$, and $\mathfrak{b}$, and under the boundary conditions given are $\mathfrak{u}(a) = \mathfrak{u}(b) = k_i$, $\mathfrak{u}'(c) = \mathfrak{u}'(d) = m_i$. When the functions involved are nonlinear, then finding the exact solution is a very difficult job. This problem originates from the formation of a numerical technique that can be used to solve the problem exactly as we required within the situation.

5. Conclusion

This study provided an introduction to obstacle problems, outlining their importance and applications. The problem was carefully formulated, extending the general model up to the third order, which broadens its scope beyond standard cases.

To support the formulation, several numerical examples were solved using Mathematica. The exact solutions obtained were consistent and reliable, confirming the validity of the approach. Alongside the analytical solutions, graphical results were also presented for each case. These visual representations not only verified the accuracy of the solutions but also highlighted a noticeable symmetry within the outcomes, reflecting the balanced structure of the underlying equations.

The findings demonstrate that the proposed framework effectively connects theoretical concepts with computational results. Moreover, the successful use of Mathematica emphasizes its usefulness in handling higher-order obstacle problems. This work establishes a solid basis for further exploration, particularly in extending the analysis to more complex settings, such as higher dimensions, nonlinear obstacles, or real-world models where such constraints naturally occur.

Author Contributions

Muhammad Amjad: Conceptualization; Haider Ali: Writing Original Draft, Review; Muntazim Abbas Hashmi: Conceptualization; Umber Rana: Validation.

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Declaration of Interest

The authors declare no conflict of interest.

Data Availability Statement

This manuscript has no associated data.

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