

Research Article

Renormalization via deformation of star products

Ali Shojaei-Fard^{1,*}

¹PhD Independent Scholar, 1461863596 Marzadaran Blvd., Tehran, Iran.

ORCID: <https://orcid.org/0000-0002-6418-3227>

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ABSTRACT

Search for non-perturbative extensions of deformation quantization is one of the most important fundamental challenges in mathematical physics and quantum theories. Applying infinite combinatorics for the construction of mathematical foundations of non-perturbative quantum field theory has already been considered and developed as an alternative practical methodology for the study of strongly coupled gauge field theories. The present article reports the impact of this new practical methodology to deformation quantization. It is shown that the topological enrichment of the Hopf algebra of renormalization leads us to generate a new family of star products which (i) encode the renormalization of combinatorial Dyson–Schwinger equations, and (ii) provide a new non-perturbative quantization program for a certain family of noncommutative algebras. Results of this study addresses a new interconnection between deformation quantization and mathematical foundations of gauge field theories in the context of the topological Hopf algebra of renormalization.

1. INTRODUCTION

Gauge field theories are mathematical platforms for the study of (interacting) quantum systems with infinite degrees of freedom. There are different approaches to formulate gauge field theories such as the renormalization Hopf algebraic setting which provided interesting interconnections between high energy physics and various disciplines of pure mathematics [10–12, 22, 31, 32, 47, 48, 55]. Perturbative renormalization and fixed point equations of (1PI) Green's functions, known as *Dyson–Schwinger equations*, have been considered in the context of the renormalization Hopf algebra of Feynman diagrams where Connes–Kreimer–Marcolli theory testifies the fundamental role of the Riemann–Hilbert correspondence in the extraction of finite values from ill-defined Feynman integrals. In addition, the Hochschild cohomology of the renormalization Hopf algebra has been considered to explain a combinatorial reformulation of Dyson–Schwinger equations [12, 33–36, 55]. At certain domains of the energy scales, the behavior of interacting quantum systems such as low energy QCD is unavailable by perturbation techniques because the number of interactions between elementary particles can tend to infinity. Formulating a topological enrichment of the renormalization Hopf algebra by a particular family of graph functions, called *stretched Feynman graphons*, led us to develop Connes–Kreimer–Marcolli theory to a non-perturbative regime for the study of non-perturbative solutions of combinatorial Dyson–Schwinger equations [49–53]. The topological Hopf algebra of renormalization, which provides a non-perturbative generalization of the Bogoliubov–

* Corresponding author. Email: shojaeifa@yahoo.com

Parasiuk–Hepp–Zimmermann (BPHZ) renormalization program [59], has already been applied as the central tool for a new construction program of the mathematical foundations of non-perturbative quantum field theory [46–48, 50, 51].

1.1 Highlighting the problem and the strategy

The combinatorics of the renormalization Hopf algebra is encoded by a certain combinatorial Hopf algebra of non-planar rooted trees [19, 31]. Polynomials of these rooted trees provide normal coordinates for the description of interaction between the BPHZ renormalization [10, 11, 59] and deformation quantization [16, 27, 30, 41, 45] underlying the Poincaré–Birkhoff–Witt basis for the renormalization Hopf algebra, the Kontsevich’s universal formula and the Baker–Campbell–Hausdorff (BCH) formula [29, 44]. On the one hand, deformation quantization of the universal enveloping of the Lie algebra of infinitesimal characters corresponding to rooted tree representations of Feynman diagrams can be described in terms of a group multiplication via the BCH formula [29, 44]. On the other hand, a basis for the combinatorial Hopf algebra of renormalization is determined by normal coordinates such that the coproduct of these normal coordinates are represented by a group multiplication and the BCH formula [19]. These investigation can lead us to search for the existence of a mutual mathematical origin for the BPHZ renormalization program and the bi-differential symplectic operator in Kontsevich’s deformation quantization under perturbative and non-perturbative settings. [29, 44]

One of the mathematical building blocks of quantum theories is formulated on the basis of the interaction between deformation quantization and quantum field theory [9, 24, 26, 40, 41]. Deformation quantization of the Poisson algebra of smooth functions on any symplectic manifold, as the phase space, enables us to pass from classical systems to quantum systems with finite degrees of freedom [1, 30, 45]. Developing this deformation quantization mechanism to the framework of quantum field theory and non-perturbative physics is the focus of ongoing research in mathematical physics community [4, 6, 17, 24, 40–42, 45, 46, 57]. Deformation theory of infinite algebras in terms of star products is a well-known way of dealing with these challenges [1, 24–26, 37, 40, 54, 58] and the present article reports a new non-perturbative deformation theory for a certain family of noncommutative algebras originated from quantum field theory. The method of deformation quantization replaces commutative algebras with noncommutative versions in terms of some quantization process on Poisson structures [30, 41, 45]. It is useful to passing from commutative space-time to a non-commutative version where coordinates find an operator nature and commutative relations between them depend on the Planck scale [6, 24, 40]. The main strategy of the present article is to formulate a deformation theory for a certain family of noncommutative algebras generated by (non)-perturbative renormalization of solutions of fixed point equations of (1PI) Green’s functions in quantum field theory. This deformation theory generates a new family of non-perturbative star products which contribute to the BPHZ renormalization program and its non-perturbative generalization.

1.2 Research background: From combinatorial Green’s functions to algebraic BPHZ renormalization

The path integral formulation of quantum field theory characterizes and classifies Green’s functions of a gauge field theory Φ in terms of N -point Green’s functions

$$G_N = \frac{\int \exp(i\frac{S[\varphi]}{\hbar})\varphi(x_1)\dots\varphi(x_N)\mathcal{D}[\varphi]}{\int \exp(i\frac{S[\varphi]}{\hbar})\mathcal{D}[\varphi]}, \quad N \geq 1, \quad N \rightarrow \infty \quad (1)$$

where $S[\varphi] = \int \mathcal{L}_\Phi[\varphi]d^Dxdt$ is the action functional with the Lagrangian density $\mathcal{L}_\Phi = \mathcal{L}_{\Phi,0} + \mathcal{L}_{\Phi,\text{int}}$ such that its interaction part $\mathcal{L}_{\Phi,\text{int}}$ is a functional of the (running) coupling constants c of the physical theory. The measure $\mathcal{D}$ is the infinite dimensional Lebesgue–Feynman measure on the space of all possible continuous paths between space-time points such that an elementary particle, as the excitation of the field φ , could travel from an initial position x_{in} at the initial state s_{in} to the final position x_{fi} at the final state s_{fi} . The interaction part $\mathcal{L}_{\Phi,\text{int}}$ contains polynomials of the field φ together with the (running) coupling constant as their coefficients. For the Lagrangian $L_\Phi = \int d^Dx \mathcal{L}_\Phi$ with the partition function

$$\mathbf{Z} = \int \exp\left(i\frac{L_\Phi[\varphi]}{\hbar}\right)\mathcal{D}[\varphi], \quad \mathbf{Z}_0 := \int \exp\left(i\frac{L_{\Phi,0}[\varphi]}{\hbar}\right)\mathcal{D}[\varphi], \quad (2)$$

Taylor expansion of $\mathbf{Z}$ around $\mathbf{Z}_0$ in the formula (1) generates a perturbative series of polynomials of powers of the running coupling constant c such that a certain family of iterated integrals on the momentum parameter, called Feynman integrals, are the coefficients of these polynomials. Each Feynman integral in these series has information about the interactions between a finite collection of elementary and virtual particles in any region of space-time. The convergence or divergence of Feynman integrals are depended on the dimension of space-time background and the domain of the momentum parameter. [12, 43]

Feynman diagrams, as a certain family of space-time diagrams, are main tools for the combinatorial representations of Green’s functions. Feynman diagrams encode and classify Feynman integrals in terms of some decorated vertices,

edges and loops. Mathematical speaking, a Feynman diagram Γ is a finite decorated oriented graph with a collection $\Gamma^{[1]} = \Gamma_{\text{int}}^{[1]} \cup \Gamma_{\text{ext}}^{[1]}$ of edges which present particles and a collection $\Gamma^{[0]}$ of vertices which present interactions between particles in the physical theory. Elements in $\Gamma_{\text{int}}^{[1]}$, which have beginning and ending vertices, are called internal edges and they are symbols for virtual particles with assigned momenta. Elements in $\Gamma_{\text{ext}}^{[1]}$, which have beginning or ending vertex, are called external edges and they are symbols for elementary particles with assigned momenta. The conservation of momenta is valid in the whole graph which means that the sums of the input momenta are the same as the sums of out momenta in each vertex and in total. Feynman rules of the physical theory provides a dictionary between Feynman diagrams and their corresponding integrals such that sub-divergences are represented by some nested loops. Superficial degree of divergence encapsulates all required factors to check whether any Feynman integral is divergent or convergent in a gauge field theory. We call Γ one particle irreducible (1PI) whenever Γ remains a connected graph after the removal of an arbitrary internal edge from it. Each subgraph in a Feynman diagram is not necessarily a Feynman subgraph and the residue operator res , which shrinks all internal edges of a graph into a single decorated vertex or a single decorated external edge, characterizes Feynman subgraphs. In other words, a Feynman subgraph γ in Γ is a graph such that $\gamma_{\text{int}}^{[1]} \subseteq \Gamma_{\text{int}}^{[1]}$, $\gamma^{[0]} \subseteq \Gamma^{[0]}$ and for each vertex $v \in \gamma^{[0]}$, $\text{res}_{\gamma}(v) = \text{res}_{\Gamma}(v)$. [12, 22, 32, 35, 43]

The N -point Green's function (1) is combinatorially represented by perturbative series of powers of the (running) coupling constant c together with higher loop order Feynman diagrams as their coefficients. For small enough values $c \ll 1$, the perturbative series is convergent but when the (running) coupling constant be strong, namely $c \geq 1$ in a certain energy scales, the perturbative series diverges such that the physical theory needs to be studied in the context of non-perturbative frameworks such as the one formulated on the basis of Dyson–Schwinger equations [31–36, 46–53].

Let $\mathcal{A}_{\Phi} = \{v_i, e_j : i, j\}$ be the finite collection of types of elementary particles e_j and types of interactions v_i between them in the physical theory Φ . 1PI Green's functions of the physical theory can be combinatorially formulated by infinite formal expansions

$$G^r = \mathbb{I} \pm \sum_{\text{res}(\Gamma)=r} \frac{c^{|\Gamma|}}{\text{Sym}(\Gamma)} \Gamma, \quad r \in \mathcal{A}_{\Phi} \quad (3)$$

such that $\mathbb{I}$ is the symbol for the empty graph and Feynman diagrams with zero loop number with the residue r , $|\Gamma|$ is the loop number of Γ , $\text{Sym}(\Gamma)$ is the rank of the automorphism group of Γ and $\text{res}(\Gamma)$ is a graph as the result of shrinking all internal edges of Γ into one vertex v_{Γ} such that external edges of Γ are attached to v_{Γ} . For $r \in \{e_j\}_j$, $\text{res}(\Gamma) = r$ means that all external edges of Γ are of type r and for $r \in \{v_i\}_i$, $\text{res}(\Gamma) = r$ means that the vertex v_{Γ} is of type r . Thanks to Feynman rules of the physical theory, for any Feynman integral $\int K_{\Gamma_0}$ corresponding to some 1PI Feynman diagram Γ_0 with $\text{res}(\Gamma_0) = r$, the equation

$$\mathbf{G}^r = 1 \pm \int K_{\Gamma_0} \mathbf{G}^r \quad (4)$$

is the original integral presentation of the Green's function of the type r . This relation shows the self-similar nature of Green's functions such that by defining $\mathcal{G}_{\Gamma_0}^r := 1 \pm \int K_{\Gamma_0} \mathbf{G}^r$, the equation (4) finds the form of a fixed point equation $\mathbf{G}^r = \mathcal{G}_{\Gamma_0}^r(\mathbf{G}^r)$. [32, 33, 35]

The method of Renormalization Group is a basic mathematical machinery for the study of changes of physical systems under different energy scales [43]. Perturbative renormalization can be described as a computational functional for the extraction of finite values from ill-defined Feynman integrals. This functional acts at two steps. The first step contains using regularization parameter such as dimensional regularization or momentum cutoff to replace Feynman integrals by some Laurent series with finite pole parts. The second step contains using minimal subtraction map to extract finite pole parts of Laurent series to generate some finite values. This functional should be applied for each nested divergent sub-integral. The (Bogoliubov–)Zimmermann's forest formula encapsulates these multiple times perturbative renormalization processes into an inductive step by step framework known as the perturbative BPHZ renormalization [59]. The combinatorics of the (Bogoliubov–)Zimmermann's forest formula were investigated on the basis of the theory of Hopf algebras where a certain coproduct structure on Feynman diagrams of the physical theory encodes the perturbative BPHZ renormalization [22, 31, 32]. This coproduct structure were developed to a graded Hopf algebra on the polynomial algebra of 1PI Feynman diagrams where the Connes–Kreimer theory interprets the BPHZ renormalization in the context of the Riemann–Hilbert problem and the Hopf–Birkhoff factorization in Rota–Baxter algebras [10, 11].

Theorem 1. *The Connes–Kreimer theory assigns a Rota–Baxter noncommutative algebra to the BPHZ renormalization.*

Proof. The renormalization coproduct is given by

$$\Delta_{\text{FG}}(\Gamma) = \mathbb{I} \otimes \Gamma + \Gamma \otimes \mathbb{I} + \sum_{\gamma} \gamma \otimes \Gamma/\gamma \quad (5)$$

for any Feynman diagram Γ in a gauge field theory Φ . The sum in the formula (5) is over all disjoint unions of non-trivial 1PI Feynman subgraphs of Γ which have some sub-divergences. This coproduct defines a connected graded finite type commutative non-commutative Hopf algebra $H_{\text{FG}}(\Phi) = \bigoplus_{n \geq 0} H^{(n)}(\Phi)$ over the field $\mathbb{Q}$ (or any field with characteristic zero) on the space of Feynman diagrams of the physical theory. It is called Connes–Kreimer renormalization Hopf algebra such that for each n , $H^{(n)}(\Phi)$ is the vector space generated by 1PI Feynman diagrams with the loop number n or products of 1PI Feynman diagrams with the overall loop number n . [10, 22]

The graded dual of $H_{\text{FG}}(\Phi)$ determines the complex Lie group $\mathbb{G}_{\Phi}(\mathbb{C}) = \text{Hom}(H_{\text{FG}}(\Phi), \mathbb{C})$ of characters on $H_{\text{FG}}(\Phi)$. The convolution product $*$, which provides the group structure, and the corresponding inverse of elements are given by

$$\psi_1 * \psi_2(\Gamma) = \psi_1 \otimes \psi_2(\Delta_{\text{FG}}(\Gamma)) = \sum_{\gamma} \psi_1(\gamma) \psi_2(\Gamma/\gamma), \quad \psi^{*-1} = \psi \circ S, \quad (6)$$

such that $S(\Gamma) = -\Gamma - \sum_{\gamma} S(\gamma) \Gamma/\gamma$ is the antipode of $H_{\text{FG}}(\Phi)$. The Lie structure on $\mathbb{G}_{\Phi}(\mathbb{C})$ is given by the topology of the pointwise convergence. In other words, the sequence $\{\phi_m\}_{m \geq 1}$ in $\mathbb{G}_{\Phi}(\mathbb{C})$ converges to character ϕ when n tends to infinity iff for any Feynman diagram Γ , $\phi_m(\Gamma)$ tends to $\phi(\Gamma)$. [10, 12]

Let $(A_{\text{dr}}, R_{\text{ms}})$ be the Rota–Baxter algebra with A_{dr} as the dimensional regularization algebra of Laurent series with finite pole parts, and $R_{\text{ms}} : A_{\text{dr}} \rightarrow A_{\text{dr}}$ as the minimal subtraction map which projects each series onto its pole parts i.e. $\sum_{n \geq -m}^{\infty} a_n z^n \mapsto \sum_{n \geq -m}^{-1} a_n z^n$. The Atkinson Theorem [19] guarantees that A_{dr} has the subdirect Birkhoff factorization $(R_{\text{ms}}(A_{\text{dr}}), -\tilde{R}_{\text{ms}}(A_{\text{dr}})) \subset A_{\text{dr}} \times A_{\text{dr}}$ with $\tilde{R}_{\text{ms}} = \text{id}_{A_{\text{dr}}} - R_{\text{ms}}$ such that each $x \in A_{\text{dr}}$ has a unique factorization $x = R_{\text{ms}}(x) + \tilde{R}_{\text{ms}}(x)$ [19]. This leads us to the existence of a unique Hopf–Birkhoff factorization for elements of the Lie algebra

$$\mathcal{L}(H_{\text{FG}}(\Phi), A_{\text{dr}}) := \quad (7)$$

$$\left\{ Z : H_{\text{FG}}(\Phi) \rightarrow A_{\text{dr}} : \mathbb{Q}\text{-linear}, Z(\Gamma_1 \Gamma_2) = Z(\Gamma_1) \varepsilon(\Gamma_2) + \varepsilon(\Gamma_1) Z(\Gamma_2) \right\}$$

of infinitesimal characters such that ε is the counit of $H_{\text{FG}}(\Phi)$. In other words, $Z = \mathcal{R}_{\text{ms}}(Z) + \tilde{\mathcal{R}}_{\text{ms}}(Z)$ such that

$$\mathcal{R}_{\text{ms}} : L(H_{\text{FG}}(\Phi), A_{\text{dr}}) \rightarrow L(H_{\text{FG}}(\Phi), A_{\text{dr}}), \quad \psi \mapsto R_{\text{ms}} \circ \psi. \quad (8)$$

The Lie algebra structure on $\mathcal{L}(H_{\text{FG}}(\Phi), A_{\text{dr}})$ is given by the commutator with respect to the convolution product $*$ given by (6). For each $\Gamma \in H_{\text{FG}}(\Phi)$, its corresponding infinitesimal character $Z_{\Gamma} : H_{\text{FG}}(\Phi) \rightarrow \mathbb{C}$ is given by $Z_{\Gamma}(\Gamma') = \delta_{\Gamma, \Gamma'}$. The collection $\{Z_{\Gamma} : \Gamma \in H_{\text{FG}}(\Phi)\}$ generates the Lie algebra $\mathcal{L}(H_{\text{FG}}(\Phi), A_{\text{dr}})$ such that

$$[Z_{\Gamma}, Z_{\Gamma'}] = Z_{\Gamma} * Z_{\Gamma'} - Z_{\Gamma'} * Z_{\Gamma} = \sum_{\Gamma'' \in H_{\text{FG}}(\Phi)} \left(n(\Gamma, \Gamma'; \Gamma'') - n(\Gamma', \Gamma; \Gamma'') \right) Z_{\Gamma''} \quad (9)$$

where $n(\Gamma, \Gamma'; \Gamma'')$ is the number of ways a subgraph Γ' can be reduced (by shrinking it to its residue) to a vertex in Γ'' such that Γ is obtained. In this case, we have $|\Gamma''| = |\Gamma| + |\Gamma'|$, $\text{res}(\Gamma'') = \text{res}(\Gamma)$ and $\text{res}(\Gamma \star \Gamma') = \text{res}(\Gamma)$ such that $\star$ is the insertion operator which defines a graded Lie algebra structure on the space of Feynman diagrams. [10, 19, 22, 31, 32, 39]

These infinitesimal characters determine the generators of the complex Lie group $\mathbb{G}_{\Phi}(A_{\text{dr}})$ of dimensionally regularized characters. Let ϕ be the one character which recovers Feynman rules of the physical theory and $z \in \partial \Delta^*$, as a complex value in the boundary of an infinitesimal punctured disk Δ^* around zero in the complex plane, be the dimensional regularization parameter. The map $\Gamma \mapsto \phi^z(\Gamma)$ sends the $(D+1)$ -dimensional Feynman integral I_{Γ} to its corresponding $(D+1-z)$ -dimensional integral I_{Γ}^z which can be presented by a Laurent series with finite pole parts. Each dimensionally regularized character $\phi^z : H_{\text{FG}}(\Phi) \rightarrow A_{\text{dr}}$ has a unique Birkhoff factorization (ϕ_-^z, ϕ_+^z) such that $\phi^z = (\phi_-^z)^{*-1} * \phi_+^z$ with $\phi_-^z \in \mathbb{G}_{\Phi}(\mathbb{C}[z^{-1}])$ and $\phi_+^z \in \mathbb{G}_{\Phi}(z^{-1}\mathbb{C}[z^{-1}])$. The renormalization coproduct (5) determines the recursive machinery

$$\phi_-^z(\Gamma) = -R_{\text{ms}}\left(\phi^z(\Gamma) + \sum_{\gamma} \phi_-^z(\gamma) \phi^z(\Gamma/\gamma)\right), \quad \phi_+^z(\Gamma) = \phi^z(\Gamma) + \sum_{\gamma} \phi_-^z(\gamma) \phi^z(\Gamma/\gamma) \quad (10)$$

for the computation of the counterterm $\phi_-^z(\Gamma)$ and the renormalized value $\phi_+^z(\Gamma)$ which are in the correspondence to the BPHZ perturbative renormalization of Γ . [10–12]

Symmetries of abelian / non-abelian gauge field theory are recovered by Ward–Takahashi / Slavnov–Taylor identities as fundamental identities between Feynman diagrams of the physical theory. These identities are encapsulated by graded Hopf ideals I_{WT} and I_{ST} for abelian and non-abelian gauge field theories. The quotient Hopf algebras $H_{\text{FG}}(\text{QED})/I_{\text{WT}}$ and $H_{\text{FG}}(\text{QCD})/I_{\text{ST}}$ restore the compatibility of quantum gauge symmetries with the Hopf algebraic reformulation of the BPHZ renormalization. [33, 35, 56] $\square$

1.3 Research methodology: Asymptotics of combinatorial Green's functions via Feynman graphon models

Theory of graphons is the result of combining graph theory in combinatorics and measure theory in analysis. Graphons and their generalized versions, called *stretched graphons*, are useful tools to represent graph limits of sequences of finite weighted dense or sparse graphs with respect to a particular family of metrics called *cut-distance* and its L^p -extensions for $p \geq 1$. Homomorphism densities, which act as continuous functionals on the space of (stretched) graphons, play an important role for the classification of the behavior of finite graphs with respect to graph limits. Theory of graphons was introduced on the Lebesgue measure space $([0, 1], m)$ and then it has been developed to arbitrary σ -finite algebras to recover graph limits of sequences of sparse graphs. [2, 3, 5, 23, 28, 38]

For the Lebesgue measure space $([0, 1], m)$, let $\mathcal{W}^{[0,1]}(\mathbb{R})$ be the topological space of bounded $\mathbb{R}$ -valued symmetric Lebesgue measurable functions on $[0, 1] \times [0, 1]$, called *graphons*, with respect to the semi-metric

$$d_{\text{cut}}(W_1, W_2) = \inf_{\rho_1, \rho_2} \sup_{A, B \subset [0, 1]} \left| \int_{A \times B} (W_1(\rho_1(x), \rho_1(y)) - W_2(\rho_2(x), \rho_2(y))) dx dy \right| \quad (11)$$

such that ρ_1, ρ_2 are invertible Lebesgue measure preserving transformations on $[0, 1]$, which generate relabelings, and A, B are Lebesgue measurable subsets in $[0, 1]$. Invertible Lebesgue measure preserving transformations on $[0, 1]$ provide infinite ways of relabeling vertices and the infimum in the formula (11) represents the best possible alignment of two graphons generated by shuffling their labels optimally to determine the smallest possible distance after checking all possible relabelings. Graphons W_1, W_2 are called *weakly isomorphic* (i.e. $W_1 \approx W_2$) iff there exist Lebesgue measure preserving transformations τ_1, τ_2 on $[0, 1]$ such that $W_1^{\tau_1} = W_2^{\tau_2}$ almost everywhere with respect to the Lebesgue measure m . The weakly isomorphic is an equivalence relation such that the semi-metric (11) becomes a metric, called *cut-distance metric*, on the quotient space $\mathcal{W}_{\approx}^{[0,1]}(\mathbb{R})$ to change it to a complete Hausdorff separable metric space which topologically completes the space of finite weighted graphs. The equivalence class $[W]_{\approx}$ is called *unlabeled graphon*. [28]

The graph limit of sequences of sparse graphs with increasing vertex set tends to the zero graphon with respect to the cut-distance metric. It is possible to assign non-zero graph limits to this family of sequences by replacing the measure space $([0, 1], m)$ with a suitable σ -finite measure space $(\Omega \subseteq [0, \infty), \mu)$ or rescaling methods such as $W \mapsto \frac{W}{\|W\|_{\text{cut}}}$ and $W \mapsto \frac{W}{\|W\|_p}$ with respect to L^p -norms [2, 3, 5]. Theory of graphons has already been modified to deal with sequences of Feynman diagrams where it is possible to assign a certain family of graphons, called *stretched Feynman graphons*, to perturbative series of Feynman diagrams which contribute to the structure of full (1PI) Green's functions. Stretched Feynman graphons present large Feynman diagrams which contribute to graph limits of sequences of higher loop order Feynman diagrams. Stretched Feynman graphons led us to study the asymptotics of (1PI) Green's functions and formulate a non-perturbative generalization of the BPHZ framework for the renormalization of fixed point equations of (1PI) Green's functions. [47–53]

Each Feynman diagram Γ has a tree (or forest) representation t_{Γ} with the adjacent matrix $\text{Ad}_{t_{\Gamma}}$. The loop number of Γ identifies the number of vertices of t_{Γ} where each vertex v_{γ} in t_{Γ} is decorated by a (1PI) primitive Feynman subgraph γ of Γ . For 1PI primitive Feynman subgraphs γ_1, γ_2 of Γ , if $\gamma_1 \subseteq \gamma_2$, then there exists an edge from the vertex v_{γ_1} to the vertex v_{γ_2} in t_{Γ} . If $\gamma_1 \not\subseteq \gamma_2$ and $\gamma_2 \not\subseteq \gamma_1$, then there is no edge between vertices v_{γ_1} and v_{γ_2} . If Γ has overlapping loops, then t_{Γ} is a linear combination of decorated non-planar rooted trees. The renormalization Hopf algebra $H_{\text{FG}}(\Phi)$ can be embedded into a decorated version of the Connes–Kreimer Hopf algebra of non-planar rooted trees $H_{\text{CK}}(\Phi)$ such that its elements are decorated by (1PI) primitive Feynman diagrams of the physical theory Φ . [19, 21, 31, 32]

Definition 1. Let $([0, \infty), \mu)$ be a σ -finite measure space and Γ any finite Feynman diagram.

- Suppose Γ has no overlapping loops, $|\Gamma| = n$ and $\{v_1, \dots, v_n\}$ is the vertex set of t_{Γ} such that each vertex v_i has a weight $\alpha_i > 0$ determined by some physical information of γ_i . Set $\alpha_{\Gamma} = \sum_{i=1}^n \alpha_i$ and define a partition $\pi = (I_1, \dots, I_n)$ in $[0, \infty)$ such that for each $1 \leq i \leq n$, $\mu(I_i) = \frac{\alpha_i}{\alpha_{\Gamma}}$ and $I_k \cap I_l = \emptyset$ for $k \neq l$. Consider the μ -measurable bounded symmetric function

$$P_{t_{\Gamma}}^{\pi} : \sqcup_{i=1}^n I_i \times \sqcup_{i=1}^n I_i \rightarrow \mathbb{R}, (x, y) \in I_n \times I_m \mapsto a_{nm} \in \text{Ad}_{t_{\Gamma}} \quad (12)$$

to define the weakly isomorphic equivalence class

$$W_{\Gamma} := [P_{t_{\Gamma}}^{\pi}]_{\approx} = \left\{ W^{\rho} : [0, \infty) \times [0, \infty) \rightarrow \mathbb{R}, P_{t_{\Gamma}} \approx W \right\} \quad (13)$$

as the unique unlabeled stretched Feynman graphon associated to Γ . In this formula, ρ is any invertible μ -measure preserving transformations on $[0, \infty)$. And $W^{\rho}(x, y) = W(\rho(x), \rho(y))$ is any μ -measurable bounded symmetric function which is weakly isomorphic to $P_{t_{\Gamma}}^{\pi}$, which means that there exists μ -measure preserving transformations ρ_1, ρ_2 on $[0, \infty)$ such that $(W^{\rho})^{\rho_1} = (P_{t_{\Gamma}}^{\pi})^{\rho_2}$ almost everywhere with respect to the measure μ .

- Suppose Γ has overlapping loops with $|\Gamma| = n$ and $t_\Gamma = c_1 s_1 + \dots + c_n s_n$ such that $|s_j| = n$ for each $1 \leq j \leq n$. Consider the unlabeled stretched graphons $W_{s_j} : C_j \times C_j \rightarrow \mathbb{R}$ associated to decorated non-planar rooted trees s_j , $1 \leq j \leq n$, of weights $\mu(C_j) = \frac{c_j}{\sum_{k=1}^n |c_k|}$ for $C_j \subset [0, \infty)$, $c_j > 0$ and $-W_{s_l} : C_l \times C_l \rightarrow \mathbb{R}$ of weights $\mu(C_l) = \frac{|c_l|}{\sum_{k=1}^n |c_k|}$ for $C_l \subset [0, \infty)$, $c_l < 0$ such that $C_r \cap C_s = \emptyset$ for $r \neq s$. The unlabeled stretched Feynman graphon W_Γ associated to Γ is defined as the direct sum of the unlabeled stretched graphons $\pm W_{s_j}$ such that $W_\Gamma|_{C_k} = \pm W_{s_k}$ and $W_{s_k} = [P_{s_k}^\pi]_\approx$.

For any σ -finite measure space $([0, \infty), \mu)$, the space $(\mathcal{W}_\approx^{[0, \infty), \mu}(\mathbb{R}))$ of stretched graphons is a complete metric space with respect to the cut-distance metric. Consider the completed metric subspace $\mathcal{S}_{\text{gr}, \approx}^{\Phi, [0, \infty), \mu}(\mathbb{R})$ of unlabeled stretched Feynman graphons associated to Feynman diagrams of the physical theory Φ . For $p \geq 1$, any $W_\Gamma \in \mathcal{S}_{\text{gr}, \approx}^{\Phi, [0, \infty), \mu}(\mathbb{R})$ is called L^p -stretched Feynman graphon if

$$\|W_\Gamma\|_{p, \text{cut}} = \sup_{A, B \subset [0, \infty)} \left(\int_{A \times B} |W_\Gamma(x, y)|^p d\mu(x) d\mu(y) \right)^{1/p} < \infty \quad (14)$$

such that the supremum is over all μ -measurable subsets A, B of $[0, \infty)$.

Theorem 2. For a gauge field theory Φ , any σ -finite measure space $([0, \infty), \mu)$ and $p \geq 1$, the metric subspace $\mathcal{S}_{\text{gr}, \approx}^{\Phi, [0, \infty), \mu}(\mathbb{R})$ of L^p -stretched Feynman graphons topologically completes the space of 1PI Green's functions of the physical theory.

Proof. This is a result of [7, 8, 47, 48, 51, 52]. The L^p -norm (14) defines the distance function on the space of Feynman diagrams

$$d_{p, \text{cut}}(\Gamma_1, \Gamma_2) := d_{p, \text{cut}}(V_{\Gamma_1}, V_{\Gamma_2}) = \inf_{\nu_1, \nu_2} \|V_{\Gamma_1}^{\nu_1} - V_{\Gamma_2}^{\nu_2}\|_{p, \text{cut}} \quad (15)$$

such that ν_1, ν_2 are invertible μ -measure preserving transformations on $[0, \infty)$ and $V_{\Gamma_1}, V_{\Gamma_2} \in \mathcal{S}_{\text{p}, \text{gr}, \approx}^{\Phi, [0, \infty), \mu}(\mathbb{R})$.

A sequence $\{\Gamma_n\}_{n \geq 1}$ of Feynman diagrams is L^p -convergent iff the corresponding sequence $\{\frac{W_{\Gamma_n}}{\|W_{\Gamma_n}\|_{p, \text{cut}}}\}_{n \geq 1}$ in $\mathcal{S}_{\text{p}, \text{gr}, \approx}^{\Phi, [0, \infty), \mu}(\mathbb{R})$ converges to a non-zero L^p -stretched Feynman graphon W_X when n tends to infinity. The large Feynman diagram X is indeed the graph limit of the sequence $\{\Gamma_n\}_{n \geq 1}$.

For each $\ell \geq 1$, consider the partial sums

$$G_{\leq \ell}^r = \mathbb{I} \pm \sum_{\Gamma, |\Gamma| \leq \ell, \text{res}(\Gamma)=r} \frac{c^{|\Gamma|}}{\text{Sym}(\Gamma)} \Gamma, \quad r \in \mathcal{A}_\Phi \quad (16)$$

of the 1PI Green's function G^r . The L^p -stretched Feynman graphon $\tilde{W}_{G_{\leq \ell}^r}$ is defined as the direct sum of L^p -stretched Feynman graphons $\tilde{W}_\Gamma : C_\Gamma \times C_\Gamma \rightarrow \mathbb{R}$ with $m(C_\Gamma) = \frac{c^{|\Gamma|}}{\text{Sym}(\Gamma)}$ and $C_\Gamma \subset [0, \infty)$. In other words, $\tilde{W}_{G_{\leq \ell}^r} : C \times C \rightarrow \mathbb{R}$ such that

$$C = \bigsqcup_{\Gamma} C_\Gamma \subset [0, \infty), \quad \tilde{W}_{G_{\leq \ell}^r}|_{C_\Gamma} = \tilde{W}_\Gamma. \quad (17)$$

When the loop number ℓ tends to infinity, the sequence $\left\{ \frac{\tilde{W}_{G_{\leq \ell}^r}}{\|\tilde{W}_{G_{\leq \ell}^r}\|_{p, \text{cut}}} \right\}_{n \geq 1}$ is L^p -convergent to the L^p -stretched Feynman graphon $\tilde{W}_{G^r} \in \mathcal{S}_{\text{p}, \text{gr}, \approx}^{\Phi, [0, \infty), \mu}(\mathbb{R})$. $\square$

The renormalization Hopf algebraic framework provided an alternative mathematical setting for the reformulation of fixed point equations of (1PI) Green's functions in the language of a certain family of recursive Hochschild equations called *combinatorial Dyson–Schwinger equations*. Solutions of these equations are given by perturbative series of powers of running coupling constants together with higher loop order Feynman diagrams recursively generated by applying Hochschild one-cocycles B_γ^+ for any (1PI) primitive Feynman diagram γ of the physical theory [33–36]. It has been shown that the space of stretched Feynman graphons on the Lebesgue measure space $([0, \infty), m)$ recover (non-)perturbative solutions of combinatorial Dyson–Schwinger equations [48–51, 51, 53].

1.4 Main results

- We assign systems of differential equations to non-perturbative renormalization of L^p -stretched Feynman graphons in the context of the topological Hopf algebra of renormalization. This investigation leads us to formulate a universal model for non-perturbative generalization of the BPHZ renormalization.
- The structure of L^p -stretched Feynman graphons associated to solutions of combinatorial Dyson–Schwinger equations is reviewed to formulate their BPHZ (non-)perturbative renormalization from the view point of deformation quantization. This study shows the appearance of non-perturbative star products in the backbone of the non-perturbative BPHZ renormalization of solutions of fixed point equations of Green's functions in quantum field theory.

2. NON-PERTURBATIVE RENORMALIZABILITY OF L^p -STRETCHED FEYNMAN GRAPHS

The Connes–Marcolli category $\mathcal{E}^{\text{CM}}$ of flat equi-singular vector bundles, formulated on the basis of the Riemann–Hilbert correspondence, recovers the BPHZ perturbative renormalization of Feynman integrals of renormalizable gauge field theories in terms of systems of differential equations together with singularities. The category $\mathcal{E}^{\text{CM}}$ is equivalent to the category of finite dimensional representations of the universal affine group scheme $\mathbb{U}^* = \mathbb{U} \rtimes \mathbb{G}_m$ such that the multiplicative group $\mathbb{G}_m$ is the affine group scheme obtained from the Hopf algebra $\mathbb{C}[t, t^{-1}]$ with coproduct $\Delta(t) = t \otimes t$. The complex Lie group $\mathbb{U}(\mathbb{C})$ is the group of characters of the universal Hopf algebra $H_{\mathbb{U}}$ which is algebraically isomorphic to the linear space of noncommutative polynomials in variables f_n , $n \in \mathbb{N}$ with the shuffle product. The Hopf–Birkhoff factorization of elements in the complex Lie group $\mathbb{U}(A_{\text{dr}})$ with respect to the Rota–Baxter algebra $(A_{\text{dr}}, R_{\text{ms}})$ led us to a universal setting for the computation of counterterms and renormalized values. For the physical theory Φ , flat equi-singular $\mathbb{G}_{\Phi}(\mathbb{C})$ -connections on the regularization bundle are considered as objects of a neutral Tannakian category $\mathcal{E}^{\Phi}$ which is equivalent to the category of finite dimensional representations of the affine group scheme $\mathbb{G}_{\Phi}^*$ and can be embedded into $\mathcal{E}^{\text{CM}}$. This universal categorical setting has been developed to a non-perturbative regime where a neutral Tannakian subcategory is assigned to the solution space of the fixed point equation of any (1PI) Green’s function of the physical theory. [12, 55]

Definition 2. Consider a gauge field theory Φ , any σ -finite measure space $([0, \infty), \mu)$ and $p \geq 1$.

- Define $H_{p, \text{gr}}(\Phi)$ as the free commutative $\mathbb{K}$ -algebra generated by those L^p -stretched Feynman graphons in $\mathcal{S}_{p, \text{gr}, \approx}^{\Phi, [0, \infty), \mu}(\mathbb{R})$ associated to 1PI Feynman diagrams in $H_{\text{FG}}(\Phi)$. The factor of loop number of Feynman diagrams assigns a graded structure on $H_{p, \text{gr}}(\Phi)$ such that for each $n \geq 1$, $H_{p, \text{gr}}^{(n)}(\Phi)$ is the vector space generated by W_{Γ} with $\Gamma \in H^{(n)}(\Phi)$.
- For $W_{\Gamma_1}, \dots, W_{\Gamma_n} \in H_{p, \text{gr}}(\Phi)$ and some scalars $c_1, \dots, c_n$ chosen from the field $\mathbb{K}$, the combination $U := c_1 W_{\Gamma_1} + \dots + c_n W_{\Gamma_n} \in H_{p, \text{gr}}(\Phi)$ is defined as the direct sum of L^p -stretched Feynman graphons $W_{\Gamma_j} : C_j \times C_j \subset [0, \infty) \times [0, \infty) \rightarrow \mathbb{R}$ with $\mu(C_j) = c_j$ for $c_j > 0$ and L^p -stretched Feynman graphons $-W_{\Gamma_l} : C_l \times C_l \subset [0, \infty) \times [0, \infty) \rightarrow \mathbb{R}$ with $\mu(C_l) = |c_l|$ for $c_l < 0$ where $C_l \cap C_t = \emptyset$ for $l \neq t$, such that for any l , $U|_{C_l}$ is weakly isomorphic to $\pm W_{\Gamma_l}$.

Theorem 3. There exist systems of differential equations which recover the renormalization of L^p -stretched Feynman graphons in $\mathcal{S}_{p, \text{gr}, \approx}^{\Phi, [0, \infty), \mu}(\mathbb{R})$

Proof. The proof contains two steps namely, showing the renormalizability of L^p -stretched Feynman graphons and then embedding their renormalization process into the Connes–Marcolli category.

Step (i). We show that L^p -stretched Feynman graphons in $\mathcal{S}_{p, \text{gr}, \approx}^{\Phi, [0, \infty), \mu}(\mathbb{R})$ are renormalizable. The renormalization coproduct (5) can be extended to $H_{p, \text{gr}}(\Phi) = \bigoplus_{n=0}^{\infty} H_{p, \text{gr}}^{(n)}(\Phi)$ which is given by

$$\Delta(W_{\Gamma}) = W_{\mathbb{I}} \otimes W_{\Gamma} + W_{\Gamma} \otimes W_{\mathbb{I}} + \sum_{W_{\gamma}} W_{\gamma} \otimes W_{\Gamma/\gamma} \quad (18)$$

with the corresponding antipode

$$S(W_{\Gamma}) = -W_{\Gamma} - \sum_{W_{\gamma}} S(W_{\gamma}) W_{\Gamma/\gamma}, \quad S(W_{\mathbb{I}}) = W_{\mathbb{I}}. \quad (19)$$

We L^p -topologically complete this Hopf algebra by the metric (15) to obtain the topological Hopf algebra $H_{p, \text{gr}}^{\text{cut}}(\Phi)$ where the space $H_{p, \text{gr}}(\Phi) \otimes H_{p, \text{gr}}(\Phi)$ equipped with the projective cross norm is a Banach space. The operators Δ and S , as linear and bounded operators, are automatically L^p -continuous.

The graded dual of $H_{p, \text{gr}}^{\text{cut}}(\Phi)$ is the complex Lie group $\mathbb{G}_{\Phi, p, \text{gr}}(\mathbb{C}) = \text{Hom}(H_{p, \text{gr}}^{\text{cut}}(\Phi), \mathbb{C})$ of characters. For the regularization parameter z as a value in the boundary of an infinitesimal punctured disk Δ^* around zero in the complex plane, elements $\tilde{\phi} \in \mathbb{G}_{\Phi, p, \text{gr}}(\mathbb{C})$ can be replaced by their regularized versions $\tilde{\phi}^z \in \mathbb{G}_{\Phi, p, \text{gr}}(A_{\text{dr}})$ such that

$$W_{\Gamma} \in H_{p, \text{gr}}^{\text{cut}}(\Phi) \mapsto \tilde{\phi}^z(W_{\Gamma}) := \phi^z(\Gamma). \quad (20)$$

For any sequence $\{W_{\Gamma_n}\}_{n \geq 1}$ of L^p -stretched Feynman graphons which is L^p -convergent to $W \in H_{p, \text{gr}}^{\text{cut}}(\Phi)$ with respect to the metric (15), $\tilde{\phi}^z(W)$ is defined as the limit of the sequence $\{\tilde{\phi}^z(W_{\Gamma_n})\}_{n \geq 1}$ with respect to the pointwise topology when n tends to infinity. Thanks to Theorem 1, the *counterterm* and *renormalized value* corresponding to the renormalization of W are computed in terms of the limits of the sequences $\left\{ \text{count}_{\tilde{\phi}^z}(W_{\Gamma_n}) \right\}_{n \geq 1}$ and $\left\{ \text{ren}_{\tilde{\phi}^z}(W_{\Gamma_n}) \right\}_{n \geq 1}$ such that

$$\text{count}_{\tilde{\phi}^z}(W) = \lim_{n \rightarrow \infty} \text{count}_{\tilde{\phi}^z}(W_{\Gamma_n}) = \lim_{n \rightarrow \infty} -R_{\text{ms}} \left(\tilde{\phi}^z(W_{\Gamma_n}) + \sum_{W_{\gamma_n}} \text{count}_{\tilde{\phi}^z}(W_{\gamma_n}) \tilde{\phi}^z(W_{\Gamma_n/\gamma_n}) \right), \quad (21)$$

$$\text{ren}_{\tilde{\phi}^z}(W) = \left(\text{count}_{\tilde{\phi}^z} \tilde{\phi}^z \right)(W) = \lim_{n \rightarrow \infty} \left(\text{count}_{\tilde{\phi}^z} \tilde{\phi}^z \right)(W_{\Gamma_n}), \quad (22)$$

such that the convolution product $\tilde{*}$ is defined in terms of the coproduct (18).

Step (ii). We show that the renormalization underlying the topological Hopf algebra $H_{p,\text{gr}}^{\text{cut}}(\Phi)$ can be described by some objects of the Connes–Marcolli category $\mathcal{E}^{\text{CM}}$.

Thanks to [12], the category of finite dimensional representations of the affine group scheme $\mathbb{G}_{\Phi,p,\text{gr}}^*$ together with graded representation

$$\eta_{\Phi,p,\text{gr}} : \mathbb{U}(\mathbb{C}) \longrightarrow \mathbb{G}_{\Phi,p,\text{gr}}(\mathbb{C}), \quad (23)$$

determines a neutral Tannakian subcategory $\mathcal{E}_{p,\text{gr}}^{\Phi}$ of $\mathcal{E}^{\text{CM}}$. The objects of $\mathcal{E}_{p,\text{gr}}^{\Phi}$ are flat equi-singular $\mathbb{G}_{\Phi,p,\text{gr}}(\mathbb{C})$ -connections on the regularization bundle $\mathbb{G}_m \rightarrow B \rightarrow \Delta$ with respect to a principal $\mathbb{G}_m(\mathbb{C}) = \mathbb{C}^*$ -bundle B . These flat equi-singular connections introduce systems of differential equations with the general form $\mathbf{D}(f) = \omega$ with singularities with respect to the logarithmic derivative

$$\mathbf{D} : \mathbb{G}_{\Phi,p,\text{gr}}(\mathbb{C}(\{z\})) \rightarrow \Omega^1(\text{Lie} \mathbb{G}_{\Phi,p,\text{gr}}(\mathbb{C})), \quad \mathbf{D}(f) = f^{-1} df. \quad (24)$$

Solutions of these equations are counterterms $\text{count}_{\tilde{\phi}^z}(W)$ for $W \in H_{p,\text{gr}}^{\text{cut}}(\Phi)$.

For the Lie algebra $\mathcal{L}_{\mathbb{U}}$ of $\mathbb{U}(\mathbb{C})$, as the free graded Lie algebra generated by elements e_{-n} of degree $n > 0$, the element $e = \sum_{n=1}^{\infty} e_{-n} \in \mathcal{L}_{\mathbb{U}}$ defines the morphism $\mathbf{rg} : \mathbb{G}_a \rightarrow \mathbb{U}$ of affine group schemes such that $\mathbb{G}_a$ is the additive group corresponds to the Hopf algebra $\mathbb{C}[t]$ with the coproduct $\Delta(t) = t \otimes 1 + 1 \otimes t$ [12]. Following [53], the representation $\eta_{\Phi,p,\text{gr}} \circ \mathbf{rg}$ determines a particular one-parameter subgroup $\{\tilde{F}_t\}_t$ in $\mathbb{G}_{\Phi,p,\text{gr}}(\mathbb{C})$ such that for each $W \in H_{p,\text{gr}}^{\text{cut}}(\Phi)$, $\tilde{F}_t(W)$ is a polynomial in t . $\square$

Corollary 1. • *The one-parameter subgroup $\{\tilde{F}_t\}_t$, determined by Theorem 3, is the Renormalization Group which governs the renormalization of L^p -stretched Feynman graphons under changing their scales.*

- *Theorem 3 extends the universal property of the Connes–Marcolli category to non-perturbative renormalization of L^p -stretched Feynman graphons.*

3. A NEW DEFORMATION THEORY FOR NONCOMMUTATIVE ALGEBRAS

Deformation theory of algebras is a fundamental mathematical approach to relate our understanding from quantum systems, studied in terms of quantizing Poisson structures of classical systems [6, 16, 20, 25, 30, 57, 58], to gauge field theories [15, 26, 37, 40–42, 45, 54]. The renormalization Hopf algebra clarified a fundamental correlation between the BPHZ renormalization program in quantum field theory and Kontsevich’s deformation quantization in terms of the BCH formula and the Kontsevich’s bi-differential symplectic operator. It is shown that the Hopf–Birkhoff factorization of regularized unrenormalized Feynman rules characters can be interpreted as the deformation of the pointwise multiplication of some exponential functions via Kontsevich’s star product in the direction of the linear Poisson bracket [29, 44]. Theory of stretched Feynman graphons has been applied recently to study quantum integrable systems in quantum field theory where a new family of solutions for Yang–Baxter equations are determined in terms of the deformation of noncommutative algebras via Rota–Baxter / Nijenhuis operators [46]. Thanks to the renormalizability of L^p -stretched Feynman graphons explained in the previous parts of the present article, here we plan to develop this new deformation theory to a certain family of noncommutative algebras originated from the non-perturbative renormalization of solutions of fixed point equations of (1PI) Green’s functions.

3.1 Combinatorial Dyson–Schwinger equations

For a gauge field theory Φ , any σ -finite measure space $([0, \infty), \mu)$ and $p \geq 1$, each of the completed metric subspaces $\mathcal{S}_{\text{gr}, \approx}^{\Phi, [0, \infty), \mu}(\mathbb{R})$ of stretched Feynman graphons or $\mathcal{S}_{p,\text{gr}, \approx}^{\Phi, [0, \infty), \mu}(\mathbb{R})$ of L^p -stretched Feynman graphons is rich enough to topologically complete the Connes–Kreimer renormalization Hopf algebra $H_{\text{FG}}(\Phi)$ of Feynman diagrams called *topological Hopf algebra of renormalization* [47, 48, 51, 53]. This particular topological Hopf algebra and its core extension contribute to the structure of a bi-Heyting algebra of Feynman diagrams which encode subsystems and also the asymptotics of (1PI) Green’s functions [47, 48].

Fixed point equations of (1PI) Green’s functions, known as *Dyson–Schwinger equations*, are considered as quantum equations of motions, shortly called *quantum motions*, in quantum field theory. For any primitive Feynman diagram γ , the grafting operator $B_{\gamma}^+ : H_{\text{FG}}(\Phi) \rightarrow H_{\text{FG}}(\Phi)$ is a linear homogeneous operator which sends each Γ to a linear combination of Feynman diagrams determined by all possible insertion places of Γ into γ in terms of types of external edges of Γ and types of vertices of γ . The Hochschild cohomology of the renormalization Hopf algebra is formulated by the complex

$(\bigoplus_{n \geq 0} C_n, \mathbf{b})$ such that for $n = 0$, $C_0 = \mathbb{Q}$ and for $n \geq 1$, C_n is the vector space generated by linear maps from $H_{FG}(\Phi)$ to $H_{FG}(\Phi)^{\otimes n}$. The coboundary operator $\mathbf{b}$ is given by

$$\mathbf{b}T(\Gamma) := (\text{id} \otimes T)\Delta_{FG}(\Gamma) + \sum_{j=1}^n (-1)^j \Delta_j(T(\Gamma)) + (-1)^{n+1} T(\Gamma) \otimes \mathbb{I}, \quad (25)$$

such that Δ_j is Δ_{FG} on the j th component of $T(\Gamma) = (T_1(\Gamma), \dots, T_n(\Gamma))$. A certain family of the generators of the first rank Hochschild cohomology group is determined by the set $\text{Prim}(H_{FG}(\Phi))$ of primitive Feynman diagrams where we have

$$\gamma \in \text{Prim}(H_{FG}(\Phi)) \mapsto \mathbf{b}B_{\gamma}^+ = 0. \quad (26)$$

Any family $\{\gamma_n\}_{n \geq 1}$ of primitive (1PI) Feynman diagrams determines the equation

$$\text{DSE}(c) : X = \mathbb{I} + \sum_{n \geq 1} c^n \omega_n B_{\gamma_n}^+(X^{n+1}) \quad (27)$$

, called *combinatorial Dyson–Schwinger equation*, in the topological ring $H_{FG}(\Phi)[[c]]$ with respect to the n -adic topology. In the equation (27), values ω_n are some scalars and c is the running coupling constant as a function of the bare coupling constant and regularization parameter. The solution $X(c) = \sum_{n \geq 0} c^n X_n$ of the equation (27) is given by the recursive relations

$$X_n = \sum_{j=1}^n \omega_j B_{\gamma_j}^+ \left(\sum_{k_1 + \dots + k_{j+1} = n-j, k_i \geq 0} X_{k_1} \dots X_{k_{j+1}} \right) \in H_{FG}(\Phi), \quad X_0 = \mathbb{I}. \quad (28)$$

such that each X_n , which is built in terms of the applications of the grafting operators on the lower order components X_j , $j < n$, contributes to the order n of the equation $\text{DSE}(c)$. The collection $\{X_n\}_{n \geq 0}$ generates a free commutative graded connected Hopf subalgebra H_{DSE} of $H_{FG}(\Phi)$. [33–36]

Corollary 2. • *The BPHZ program on topological Hopf algebra of renormalization assigns renormalized values to the large Feynman diagram $X(c)$.*

- *The morphism $X(c) \mapsto W_{X(c)}$ defines an injective functor from the category $\mathcal{E}_{\text{p,DSE}}^{\Phi}$ to the category $\mathcal{E}_{\text{p,gr}}^{\Phi}$. Therefore $\mathcal{E}_{\text{p,gr}}^{\Phi}$, which embeds into the Connes–Marcolli category $\mathcal{E}^{\text{CM}}$, recovers a universal platform for the (non-)perturbative renormalization of solutions of combinatorial Dyson–Schwinger equations of the physical theory.*

Proof. It is a direct result of Theorems 1, 2, 3, Corollary 1 together with [12, 51, 55]. $\square$

We apply Definitions 1, 2 and Theorem 2. For the sequence $\{Y_m\}_{m \geq 1}$ of the partial sums of the solution $X(c)$, the rooted forest representations $t_{Y_m} = ct_{X_1} + \dots + c^m t_{X_m}$, $m \geq 1$, determine the sequence $\{\tilde{W}_{Y_m}\}_{m \geq 1}$ of L^p -stretched Feynman graphons such that

$$\tilde{W}_{Y_m} = c\tilde{W}_{X_1} + \dots + c^m \tilde{W}_{X_m} : \prod_{k=1}^m \tilde{J}_k \times \prod_{k=1}^m \tilde{J}_k \subseteq [0, \infty) \times [0, \infty) \rightarrow \mathbb{R} \quad (29)$$

is the direct sum L^p -stretched Feynman graphons $\tilde{W}_{X_k} : \tilde{J}_k \times \tilde{J}_k \rightarrow \mathbb{R}$ with $\mu(\tilde{J}_k) = c^k$, $1 \leq k \leq m$. We have

$$\lim_{m \rightarrow \infty} Y_m = X(c) \Leftrightarrow \lim_{m \rightarrow \infty} \frac{\tilde{W}_{Y_m}}{\|\tilde{W}_{Y_m}\|_{\text{p,cut}}} = \tilde{W}_{X(c)}, \quad L^p - \text{metric}. \quad (30)$$

While the L^p -stretched Feynman graphon $\tilde{W}_{X(c)}$ presents the large Feynman diagram $X(c)$, it opens the new possibility of studying the asymptotics of $X(c)$ in terms of estimating probability distributions of its partial sums Y_m at large orders m through sequences of correlated inductive random graphs.

Corollary 3. *The large Feynman diagram $X(c)$ has a stochastic model.*

Proof. For any order $m \geq 1$, consider the vertex set V_m of t_{Y_m} as a poset to embed it into a new poset $A_m := \bigsqcup_{k=1}^m J_k$ in $[0, \infty)$ with cardinal $|V_m|$ such that J_k is the sub-poset corresponding to the vertex set of t_{X_k} . Define a new random graph R_m with the vertex set A_m such that with the probability $\mathbf{p}_m(x, y) := |W_{Y_m}(x, y)| - |W_{Y_m}(x, y)|$ there exists an edge between x and y in R_m . When m tends to infinity, the sequence $\{R_m\}_{m \geq 1}$ converges to an infinite random graph $R_{X(c)}$ defined on the

infinite countable poset $A_\infty := \bigsqcup_{k=1}^\infty J_k$ in $[0, \infty)$ such that with the probability $\mathbf{p}_{X(c)}(x, y) := |\tilde{W}_{X(c)}(x, y)| - \left| |\tilde{W}_{X(c)}(x, y)| \right|$ there exists an edge between x and y in R_∞ .

This process of random graphs governs probability distributions of the partial sums Y_m in terms of the formula

$$\mathbb{P}_{X(c)}(Y_m) := \int_{A_m^{|V_m|}} \prod_{e_{ij} \in E(t_{Y_m})} \mathbf{p}_{X(c)}(x_i, x_j) \prod_{e_{ij} \notin E(t_{Y_m})} (1 - \mathbf{p}_{X(c)}(x_i, x_j)) dx_1 \dots dx_{|V_m|}. \quad (31)$$

□

3.2 Star products of non-perturbative renormalization

Let $H_{p,DSE}^{\text{cut}}$ be the topological Hopf subalgebra of $H_{p,FG}^{\text{cut}}(\Phi)$ corresponding to the solution $X(c)$ of the combinatorial Dyson–Schwinger equation DSE and topologically completed by the metric (15) and L^p -stretched Feynman graphons in $\mathcal{S}_{p,gr,\approx}^{\Phi,[0,\infty)\mu}(\mathbb{R})$. For a field $\mathbb{K}$ of characteristic zero, consider the convolution algebra $L(H_{p,DSE}^{\text{cut}}, A_{\text{dr}})$ of $\mathbb{K}$ -linear maps from $H_{p,DSE}^{\text{cut}}$ to A_{dr} such that the functional

$$\mathcal{R} : L(H_{p,DSE}^{\text{cut}}, A_{\text{dr}}) \rightarrow L(H_{p,DSE}^{\text{cut}}, A_{\text{dr}}), \quad \phi \mapsto R_{\text{ms}} \circ \phi, \quad (32)$$

defines Nijenhuis maps $\mathcal{R}_\lambda := \mathcal{R} - \lambda \tilde{\mathcal{R}}$ on $L(H_{p,DSE}^{\text{cut}}, A_{\text{dr}})$ for each $\lambda \in \mathbb{K}$ such that $\tilde{\mathcal{R}} := \text{Id}_{L(H_{p,DSE}^{\text{cut}}, A_{\text{dr}})} - \mathcal{R}$. We apply [18, 54] to deform the algebraic structure of $L(H_{p,DSE}^{\text{cut}}, A_{\text{dr}})$ to generate some new associative multiplications on $L(H_{p,DSE}^{\text{cut}}, A_{\text{dr}})$ given by

$$\phi \circ_\lambda \psi := \mathcal{R}_\lambda(\phi) * \psi + \phi * \mathcal{R}_\lambda(\psi) - \mathcal{R}_\lambda(\phi * \psi). \quad (33)$$

Definition 3. For each $\lambda \in \mathbb{K}$, the commutator with respect to $\circ_\lambda$, given by

$$[\phi, \psi]_\lambda := [\mathcal{R}_\lambda(\phi), \psi] + [\phi, \mathcal{R}_\lambda(\psi)] - \mathcal{R}_\lambda[\phi, \psi], \quad (34)$$

defines a new Poisson structure on the noncommutative associative unital $\mathbb{K}$ -algebra

$$C_{\lambda,p,\text{cut}}^{\text{DSE}} := (L(H_{p,DSE}^{\text{cut}}, A_{\text{dr}}), \circ_\lambda). \quad (35)$$

Corollary 4. There exists a family of star products which recover the renormalization of the large Feynman diagram $X(c)$.

Proof. We apply Theorem 3, Corollary 2 and Definition 3 together with [19, 25, 29, 44].

The space $\text{char}_\lambda H_{p,DSE}^{\text{cut}} \subset C_{\lambda,p,\text{cut}}^{\text{DSE}}$ of characters is equipped with a complex Lie group structure with the corresponding Lie algebra $\partial \text{char}_\lambda H_{p,DSE}^{\text{cut}} \subset C_{\lambda,p,\text{cut}}^{\text{DSE}}$ of infinitesimal characters. For each $n \geq 1$, consider the Hopf sub-subalgebra $H_{(n)}$ of H_{DSE} free generated by a finite number of generators $X_0, X_1, \dots, X_n$ of the solution of DSE with the corresponding infinitesimal characters $Z_{X_0}, Z_{X_1}, \dots, Z_{X_n}$. Define a new star product on $H_{(n)}$ given by

$$\phi_{(n)} \star_{[\dots]_\lambda} \psi_{(n)} = \phi_{(n)} * \psi_{(n)} + \hbar [\phi_{(n)}, \psi_{(n)}]_\lambda + O(\hbar^2), \quad (36)$$

such that $\phi_{(n)} := \phi|_{H_{(n)}}$, $\psi_{(n)} := \psi|_{H_{(n)}}$ for characters $\phi, \psi \in \text{char}_\lambda H_{\text{DSE}}^{\text{p,cut}}$. This star product can be extended under a non-perturbative setting in terms of defining $\phi \star_{[\dots]_\lambda} \psi$ as the limit of the sequence $\left\{ \phi_{(n)} \star_{[\dots]_\lambda} \psi_{(n)} \right\}_{n \geq 1}$.

Thanks to the convergence relation (30), the Hopf–Birkhoff factorization $(\tilde{\phi}_-^z, \tilde{\phi}_+^z)$ of the regularized character $\tilde{\phi}^z \in L(H_{p,DSE}^{\text{cut}}, A_{\text{dr}})$ together with relations (21) and (22), we have

$$\begin{aligned} & \left((\tilde{\phi}_-^z)^{-1} \star_{[\dots]_\lambda} \tilde{\phi}_+^z \right) (X(c)) \\ &= \left((\tilde{\phi}_-^z)^{-1} \star_{[\dots]_\lambda} \tilde{\phi}_+^z \right) (\lim_{m \rightarrow \infty} Y_m) = \lim_{m \rightarrow \infty} \left((\tilde{\phi}_{(m),-}^z)^{-1} \star_{[\dots]_\lambda} \tilde{\phi}_{(m),+}^z \right) (Y_m). \end{aligned} \quad (37)$$

This leads us to reformulate the renormalization of $X(c)$ in terms of the BCH formula for the Hausdorff series underlying $\star_{[\dots]_\lambda}$. □

The method of the Dubois-Violette’s noncommutative differential calculus underlying noncommutative deRham complex of the differential graded algebras is a well-known technique to assign geometry models to noncommutative algebras [13–15, 54] and here we apply this method to formulate a new quantization program for algebras $C_{\lambda,p,\text{cut}}^{\text{DSE}}$ in the context of the deformation of deRham complex.

Lemma 1. *There exists a bijective correspondence between $\partial \text{char}_\lambda H_{\text{p,DSE}}^{\text{cut}}$ and $\text{char}_\lambda H_{\text{p,DSE}}^{\text{cut}}$.*

Proof. For all $\Gamma \in H_{\text{p,DSE}}^{\text{cut}}$, consider infinitesimal characters Z_Γ , as generators of the Lie algebra

$$\left(L(H_{\text{p,DSE}}^{\text{cut}}, A_{\text{dr}}), [\cdot, \cdot]_\lambda \right), \quad (38)$$

which are *Hamiltonian derivations*. The exponential map is well-defined and given by

$$\partial \text{char}_\lambda H_{\text{p,DSE}}^{\text{cut}} \rightarrow \text{char}_\lambda H_{\text{p,DSE}}^{\text{cut}}, \quad \exp^{\circ_\lambda}(Z) := \sum_{n \geq 0} \frac{Z^{\circ_\lambda n}}{n!}. \quad (39)$$

It can be extended to the infinitesimal character $Z_{X(c)}$, corresponding to the large Feynman diagram $X(c)$, as the limit of the sequence $\{Z_{Y_m}\}_{m \geq 1}$ of infinitesimal characters corresponding to partial sums Y_m with respect to the pointwise topology. In other words, for any $\Gamma \in H_{\text{p,DSE}}^{\text{cut}}$, the sequence $\{Z_{Y_m}(\Gamma)\}_{m \geq 1}$ is convergent to $Z_{X(c)}(\Gamma)$ when m tends to infinity. $\square$

Corollary 5. *There exists a star product which recovers the quantization of the geometry of $C_{\lambda,\text{p,cut}}^{\text{DSE}}$.*

Proof. Let $Z(C_{\lambda,\text{p,cut}}^{\text{DSE}})$ be the center of the algebra $C_{\lambda,\text{p,cut}}^{\text{DSE}}$, $\text{Der}_{\lambda,\text{p,cut,DSE}}$ be the space of derivations $\theta : C_{\lambda,\text{p,cut}}^{\text{DSE}} \rightarrow C_{\lambda,\text{p,cut}}^{\text{DSE}}$ as linear maps which satisfy Leibniz rule. Following Definition 3, let $\text{Ham}_{\lambda,\text{p,cut,DSE}}$ be the $Z(C_{\lambda,\text{p,cut}}^{\text{DSE}})$ -module generated by Hamiltonian derivations $\text{ham}(\phi) : \psi \mapsto [\phi, \psi]_\lambda$ for all $\phi \in C_{\lambda,\text{p,cut}}^{\text{DSE}}$.

For $n \geq 1$, define $\Omega^n(C_{\lambda,\text{p,cut}}^{\text{DSE}})$ as the space of all $Z(C_{\lambda,\text{p,cut}}^{\text{DSE}})$ -multilinear antisymmetric functionals

$$\text{Ham}_{\lambda,\text{p,cut,DSE}} \times \dots \times \text{Ham}_{\lambda,\text{p,cut,DSE}} \rightarrow C_{\lambda,\text{p,cut}}^{\text{DSE}} \quad (40)$$

such that for $n = 0$, $\Omega^0(C_{\lambda,\text{p,cut}}^{\text{DSE}}) = C_{\lambda,\text{p,cut}}^{\text{DSE}}$. The exterior derivative of this differential graded algebra is given by

$$(\mathbf{d}_\lambda \omega)(\theta_0, \dots, \theta_n) := \quad (41)$$

$$\sum_{k=0}^n (-1)^k \theta_k \omega(\theta_0, \dots, \hat{\theta}_k, \dots, \theta_n) + \sum_{0 \leq r < s \leq n} (-1)^{r+s} \omega([\theta_r, \theta_s]_\lambda, \theta_0, \dots, \hat{\theta}_r, \dots, \hat{\theta}_s, \dots, \theta_n)$$

for any $\omega \in \Omega^\bullet(C_{\lambda,\text{p,cut}}^{\text{DSE}}) = \bigoplus_{n=0}^\infty \Omega^n(C_{\lambda,\text{p,cut}}^{\text{DSE}})$ and $\theta_0, \dots, \theta_n \in \text{Der}_{\lambda,\text{p,cut,DSE}}$. The resulting *noncommutative de Rham complex* $\text{DR}_{\lambda,\text{p,cut,DSE}}^\bullet := \frac{\Omega^\bullet(C_{\lambda,\text{p,cut}}^{\text{DSE}})}{[\Omega^\bullet(C_{\lambda,\text{p,cut}}^{\text{DSE}}), \Omega^\bullet(C_{\lambda,\text{p,cut}}^{\text{DSE}})]_\lambda}$ provides the required symplectic form to determine the Poisson structure of a geometry model for $C_{\lambda,\text{p,cut}}^{\text{DSE}}$. The Poisson structure is given by $\{f, g\}_\lambda = i_{\theta_f^\lambda} i_{\theta_g^\lambda} \omega_\lambda$ such that $\theta_f^\lambda, \theta_g^\lambda$ are *symplectic vector fields* associated with $\mathbf{d}_\lambda f, \mathbf{d}_\lambda g$ and for $\theta = \sum_i u_i \circ_\lambda \text{ham}(g_i)$, $\theta' = \sum_j v_j \circ_\lambda \text{ham}(h_j)$

$$\omega_\lambda : \text{Ham}_{\lambda,\text{p,cut,DSE}} \times \text{Ham}_{\lambda,\text{p,cut,DSE}} \rightarrow C_{\lambda,\text{p,cut}}^{\text{DSE}}, \quad \omega_\lambda(\theta, \theta') = \sum_{i,j} u_i \circ_\lambda v_j \circ_\lambda [g_i, h_j]_\lambda \in \Omega^2(C_{\lambda,\text{p,cut}}^{\text{DSE}}) \quad (42)$$

is the *symplectic form*.

The quantization of the Poisson structure $\{\cdot, \cdot\}_\lambda$ is defined by the new star product $\tilde{\star}_{\{\cdot, \cdot\}_\lambda}$ on $L(H_{\text{p,DSE}}^{\text{cut}}, A_{\text{dr}})$ given by

$$f \tilde{\star}_{\{\cdot, \cdot\}_\lambda} g = f * g + \hbar \{f, g\}_\lambda + O(\hbar^2). \quad (43)$$

$\square$

Corollary 6. *The star product $\tilde{\star}_{\{\cdot, \cdot\}_\lambda}$ delivers a non-perturbative quantization of the geometry of $C_{\lambda,\text{p,cut}}^{\text{DSE}}$.*

Proof. For any sequence $\{\Gamma_n\}_{n \geq 1} \subset H^{\text{p,DSE}}$ which L^p -converges to Γ with respect to the metric (15) when n tends to infinity, we get

$$(f \tilde{\star}_{\{\cdot, \cdot\}_\lambda} g)(\Gamma) := \lim_{n \rightarrow \infty} (f \tilde{\star}_{\{\cdot, \cdot\}_\lambda} g)(\Gamma_n). \quad (44)$$

In particular, for the Hopf–Birkhoff factorization $(\tilde{\phi}_-^z, \tilde{\phi}_+^z)$ of the regularized character $\tilde{\phi}^z \in L(H_{\text{p,DSE}}^{\text{cut}}, A_{\text{dr}})$ and the large Feynman diagram $X(c)$, we have

$$\begin{aligned} & \left((\tilde{\phi}_-^z)^{* -1} \tilde{\star}_{\{\cdot, \cdot\}_\lambda} \tilde{\phi}_+^z \right) (X(c)) \\ &= \left((\tilde{\phi}_-^z)^{* -1} \tilde{\star}_{\{\cdot, \cdot\}_\lambda} \tilde{\phi}_+^z \right) (\lim_{m \rightarrow \infty} Y_m) = \lim_{m \rightarrow \infty} \left((\tilde{\phi}_-^z)^{* -1} \tilde{\star}_{\{\cdot, \cdot\}_\lambda} \tilde{\phi}_+^z \right) (Y_m) \end{aligned} \quad (45)$$

with respect to the sequence $\{Y_m\}_{m \geq 1}$ of the partial sums of $X(c)$. The existence of these limits are supported by the topological completeness of $H_{\text{p,DSE}}^{\text{cut}}$ via $\mathcal{S}_{\text{p,gr}, \approx}^{\Phi, [0, \infty), \mu}(\mathbb{R})$. $\square$

4. CONTRIBUTION OF THIS STUDY TO THE PROBLEM

The original contribution of the present article to the problem highlighted in subsection 1.1 is to extend the interaction between the BPHZ renormalization and deformation quantization to a non-perturbative setting where a new family of star products on a certain family of noncommutative algebras have been constructed. These star products, which encode non-perturbative renormalization of solutions of fixed point equations of (1PI) Green's functions, enables us to formulate geometry models for noncommutative algebras where quantization of these geometries can be described by the deformation of de Rham complex. The following steps have been completed in the present article to achieve the addressed contribution.

- We discussed the topological completeness of the space of (1PI) Green's functions of any gauge field theory Φ via the metric space $\mathcal{S}_{p,gr,\approx}^{\Phi,[0,\infty),\mu}(\mathbb{R})$ of L^p -stretched Feynman graphons for $p \geq 1$, see Theorem 2. This methodology has led us to obtain a new geometric interpretation of the renormalization of L^p -stretched Feynman graphons underlying the Riemann–Hilbert correspondence, see Theorem 3 and Corollary 1. Then we studied fixed point equations of (1PI) Green's functions, reformulated in the context of the Hochschild cohomology of the renormalization Hopf algebra, underlying L^p -stretched Feynman graphon representations to assign stochastic models to their solutions, see Corollaries 2 and 3.
- We applied the Rota–Baxter property of the minimal subtraction map on the dimensional regularization algebra of Laurent series with finite pole parts to generate a new family of deformations of the convolution product (6) as the multiplications of new noncommutative algebras $C_{\lambda,p,cut}^{DSE}$. These noncommutative algebras contribute to the renormalization of solutions of fixed point equations of (1PI) Green's functions by introducing a new family of star products. In addition, it is shown that the existence of a particular family of star products which provides a non-perturbative quantization program for the geometry of the noncommutative algebras $C_{\lambda,p,cut}^{DSE}$. See Corollaries 4, 5 and 6.

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