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Research Article

Hyers–Ulam Stability for Fractional Hybrid Switched Systems in Hölder Spaces

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ABSTRACT

This paper establishes a comprehensive theory for fractional hybrid switched differential equations (FHSDEs) in Hölder spaces. We prove the existence of solutions using Schauder's fixed-point theorem, uniqueness via Banach's contraction principle, and Hyers-Ulam stability through a direct fixed-point approach. The switched system framework incorporates abrupt changes in fractional dynamics at switching points, while Hölder spaces provide the natural setting for solution regularity. Detailed proofs feature explicit calculations of key constants and inequalities. An illustrative example demonstrates the applicability of our results, including explicit stability constant computations. This work extends hybrid fractional theory to switched systems with Hölder-continuous solutions.

1. INTRODUCTION

Fractional differential equations (FDEs) have revolutionized mathematical modeling by capturing hereditary properties and memory effects inherent in complex systems across physics, engineering, and biology. The foundational work of Riemann-Liouville and Caputo fractional operators [1, 2] extended classical calculus to non-integer orders, enabling precise descriptions of anomalous diffusion, viscoelastic materials, and electrochemical processes. The fractional derivative D^ρ $(\rho \in (0,1))$ intrinsically embeds history dependence through its convolution kernel, making it indispensable for systems where current dynamics depend on past states [3, 4]. Subsequent theoretical advances established existence-uniqueness theorems [5] and stability criteria [6] for FDEs, laying the groundwork for hybrid extensions that incorporate discontinuous nonlinearities.

Hybrid fractional differential equations (HFDEs), pioneered by Dhage [7], unified fractional calculus

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with discontinuous perturbations through formulations of the form:

$$D^{\rho}\left[\frac{\kappa(t)}{\zeta(t,\kappa(t))}\right] = \psi(t,\kappa(t)). \tag{1}$$

This framework captures systems where continuous memory-dependent evolution interacts with abrupt state changes, such as biological switches [8] or impact mechanics [9]. Zhao et al. [8] established existence results via fixed-point theorems, while Benchohra [10] developed stability theory under Carathéodory conditions. Parallel advances in switched systems theory addressed modern engineering challenges where discrete transitions between subsystems govern global behavior. Sun and Ge [11] established stability criteria for switched ordinary differential equations (ODEs), while Liberzon [12] developed foundational Lie-algebraic conditions for stability under arbitrary switching. Applications span power electronics [13] and biochemical regulation [14].
The synthesis of these paradigms, fractional hybrid systems and switched dynamics,yields *fractional hybrid switched differential equations* (FHSDEs):

$$D^{\rho_{\sigma(t)}}\left[\frac{\kappa(t)}{\zeta_{\sigma(t)}(t,\kappa(t))}\right] = \psi_{\sigma(t)}(t,\kappa(t)), \quad \kappa(0)=0, \quad t\in[0,T], \tag{2}$$

where $\sigma(t) : [0,T] \to 1,\ldots,N$ is a piecewise constant switching signal. This framework models systems with *memory-dependent mode transitions*, such as:

- Power grids with fractional-order controllers switching between stability regimes [15]
- Neurobiological processes exhibiting threshold-triggered memory effects [16]
- Robotic systems with impact dynamics governed by viscoelastic contacts [17].

Despite their modeling power, FHSDEs lack comprehensive theoretical foundations. Existing results focus primarily on non-switched hybrid systems [18] or switched FDEs without hybrid structure [19]. Key gaps include:

1. Existence-uniqueness theory in function spaces accommodating solution regularity at switching points
2. Quantitative stability bounds for robust control design
3. Compatibility conditions ensuring solution regularity across switching instants

Hölder spaces $H^{\varsigma}[0,T]$ ($\varsigma \in (0,1]$) provide the natural setting for FHSDE analysis. Fractional operators D^{ρ} map H^{ς} to $H^{\varsigma-\rho}$ [20], and solutions typically exhibit Hölder continuity due to the kernel singularity at switching points [21]. While Kilbas et al. [2] established Hölder regularity for linear FDEs, hybrid switched systems introduce three challenges:

1. *Switching compatibility*: Solutions must satisfy continuity conditions at t_k despite fractional-order jumps
2. *Nonlinear coupling*: Hybrid terms $\zeta_i(t,\kappa)$ require Lipschitz conditions compatible with Hölder norms
3. *Stability constant computation*: Explicit HU bounds depend on sharp norm estimates

Hyers-Ulam (HU) stability, originating in functional equations [22], ensures that ϵ-approximate solutions remain near exact solutions, a critical property for systems with modeling uncertainties. Alsina and Ger [23] established HU stability for ODEs, while Castro and Ramos [24] extended it to fractional systems. Recent advances include:

- Devi and Kumar's HU results for hybrid fractional equations with p-Laplacian [18]
- Bhujel et al.'s stability bounds for hybrid FDEs [25]
- Wang et al.'s stability criteria for impulsive switched FDEs [15]

However, no existing work addresses HU stability for FHSDEs in Hölder spaces, and explicit stability constants remain uncomputed for switched hybrid systems.
This paper bridges these gaps through four key contributions:

1. *Existence theory*: Using Schauder's fixed-point theorem in $H^{\varsigma}[0,T]$, we prove solution existence under Carathéodory-type conditions (Theorem 3), with compatibility conditions ensuring continuity at switching points (Lemma 1).
2. *Uniqueness*: Via Banach's contraction principle with sharp Lipschitz constants (Theorem 4), we establish solution uniqueness and derive the contractivity parameter C_3 governing stability bounds.
3. *HU stability*: We prove FHSDEs are Hyers-Ulam stable with explicit constant $K = (1 - C_3)^{-1}\max(1, T^{\varsigma})$ (Theorem 5), providing computable robustness margins.
4. *Switching compatibility*: Detailed norm estimates (Proposition 1) ensure solutions in H^{ς} remain regular across switching instants, resolving a critical challenge in hybrid fractional theory.

Our illustrative example (Sec. 5) computes $C_3 \approx 0.396$ and $K \approx 1.657$, validating the theory with numerically verifiable constants. This work establishes Hölder spaces as the natural functional setting for FHSDE analysis while providing tools for robust control of memory-dependent switched systems.

Paper organization. Section 2 introduces notation, Hölder spaces, and fractional calculus. Section 3 proves solution existence via Schauder's theorem. Section 4 establishes uniqueness and contraction estimates. Section 5 derives HU stability bounds and presents the computational example. Section 6 discusses implications and future directions

2. NOTATION AND PRELIMINARIES

We denote by $\mathbb{R}^+$ the set of non-negative real numbers, $[0,T]$ a compact time interval, and $\Gamma(\cdot)$ the Gamma function. The $L^1[0,T]$ norm is $||h||_{L^1} = \int_0^T |h(s)|ds$. Hölder spaces $H^{\varsigma}[0,T]$ ($\varsigma \in (0,1]$) consist of functions satisfying:

$$|\kappa(t) - \kappa(s)| \leq H_\kappa |t-s|^{\varsigma} \quad \forall t, s \in [0,T]$$

with norm $||\kappa||_{\varsigma} = |\kappa(0)| + H_\kappa$ where $H_\kappa = \sup_{t \neq s} \frac{|\kappa(t)-\kappa(s)|}{|t-s|^{\varsigma}}$ is the Hölder seminorm. The supremum norm is $||\kappa||_\infty = \sup_{t\in[0,T]} |\kappa(t)|$.

Proposition 1. *For* $\kappa \in H^{\varsigma}[0,T]$ *and* $0 < \varsigma \leq 1$*:*

$$||\kappa||_\infty \leq \max(1, T^{\varsigma})\, ||\kappa||_{\varsigma}$$

Proof. Direct computation: $|\kappa(t)| \le |\kappa(0)| + H_\kappa t^\varsigma \le |\kappa(0)| + H_\kappa T^\varsigma \le \max(1, T^\varsigma)\|\kappa\|_\varsigma$.

The Riemann-Liouville integral of order $\rho > 0$ is:

$$(I^\rho \kappa)(t) = \frac{1}{\Gamma(\rho)} \int_0^t (t-s)^{\rho-1}\kappa(s)ds$$

and the Riemann-Liouville derivative of order $\rho \in (0,1)$ is:

$$(D^\rho \kappa)(t) = \frac{d}{dt} I^{1-\rho}\kappa(t).$$

Lemma 1 (Equivalence Lemma)**.** *The FHSDE* (2) *is equivalent to:*

$$\kappa(t) = \zeta_{\sigma(t)}(t, \kappa(t)) \cdot I^{\rho_{\sigma(t)}}\psi_{\sigma(t)}(t, \kappa(t)), \quad t \in [0, T] \tag{3}$$

Remark 1. *At switching points t_k, solutions satisfy* $\lim_{t\to t_k^-} \kappa(t) = \lim_{t\to t_k^+} \kappa(t)$ *due to the continuity requirement in the equivalence.*

Proof. **Part 1: Differential equation implies integral equation.** Consider $t \in [t_k, t_{k+1})$ with $\sigma(t) = i$ constant. Define:

$$g(t) = \frac{\kappa(t)}{\zeta_i(t, \kappa(t))}.$$

The equation becomes:

$$D^{\rho_i} g(t) = \psi_i(t, \kappa(t)).$$

Apply I^{ρ_i} to both sides:

$$I^{\rho_i} D^{\rho_i} g(t) = I^{\rho_i}\psi_i(t, \kappa(t)).$$

By the fractional fundamental theorem:

$$I^{\rho_i} D^{\rho_i} g(t) = g(t) - \frac{t^{\rho_i - 1}}{\Gamma(\rho_i)} \lim_{\tau\to 0^+} I^{1-\rho_i} g(\tau).$$

Since $g(0) = \kappa(0)/\zeta_i(0, \kappa(0)) = 0$ and g is continuous at 0, the limit $\lim_{\tau\to 0^+} I^{1-\rho_i} g(\tau) = 0$. Thus:

$$g(t) = I^{\rho_i}\psi_i(t, \kappa(t)),$$

which implies:

$$\kappa(t) = \zeta_i(t, \kappa(t)) \cdot I^{\rho_i}\psi_i(t, \kappa(t)).$$

Continuity at switching points follows from the continuity of the right-hand side.
Part 2: Integral equation implies differential equation. Assume (3) holds. Define:

$$g(t) = \frac{\kappa(t)}{\zeta_i(t, \kappa(t))} = I^{\rho_i}\psi_i(t, \kappa(t)).$$

Apply D^{ρ_i}:

$$D^{\rho_i} g(t) = D^{\rho_i} I^{\rho_i}\psi_i(t, \kappa(t)) = \psi_i(t, \kappa(t)).$$

The initial condition $\kappa(0) = 0$ holds by construction.

We recall fixed-point theorems:

Theorem 1 (Schauder)**.** *If S is a nonempty, closed, convex subset of a Banach space and $T : S \to S$ is compact and continuous, then T has a fixed point.*

Theorem 2 (Banach)**.** *Every contraction on a complete metric space has a unique fixed point.*

3. EXISTENCE RESULTS

Building upon the Hölder space framework and fractional calculus foundations established in Section 2, we now prove the existence of solutions for fractional hybrid switched differential equations. Leveraging the compact embedding property of Hölder spaces theorem and Schauder's fixed-point theorem, we develop sufficient conditions under which solutions exist despite switching dynamics. The following assumptions bridge our preliminary framework to the solution existence theory.

(A1) The map $\kappa \mapsto \kappa/\zeta_i(t,\kappa)$ is strictly increasing for each i.

(A2) There exist $a_i > 0$ and continuous nondecreasing $\phi_i : \mathbb{R}^+ \to \mathbb{R}^+$ such that:

$$|\zeta_i(t,\kappa) - \zeta_i(s,y)| \leq a_i|t-s| + \phi_i(|\kappa - y|)$$

(A3) There exist $G_i : \mathbb{R}^+ \to \mathbb{R}^+$ continuous nondecreasing and $h_i \in L^1[0,T]$ such that:

$$|\psi_i(t,\kappa) - \psi_i(t,y)| \leq G_i(|\kappa - y|), \quad |\psi_i(t,\kappa)| \leq h_i(t)$$

(A4) There exists $r_0 > 0$ satisfying for all i:

$$\frac{1}{\Gamma(\rho_i)}\left(\phi_i(r_0) + \bar{\zeta}_i\right)\|h_i\|_{L^1} T^{\rho_i - 1} + \frac{1}{\Gamma(\rho_i+1)}\left(a_i T^{1-\min(\varsigma,\rho_i)} + \phi_i(r_0)\right)\|h_i\|_{L^1} \leq r_0 \tag{4}$$

where $\bar{\zeta}_i = \sup_{t\in[0,T]} |\zeta_i(t,0)|$.

Theorem 3. *Under (A1)-(A4), FHSDE* (2) *has a solution in $H^{\varsigma}[0,T]$ with $\varsigma \leq \min_i \rho_i$.*

Proof. Define the operator $\mathcal{T} : H^{\varsigma}[0,T] \to H^{\varsigma}[0,T]$ by:

$$(\mathcal{T}\kappa)(t) = \zeta_{\sigma(t)}(t,\kappa(t)) \cdot I^{\rho_{\sigma(t)}}\psi_{\sigma(t)}(t,\kappa(t)).$$

To show Hölder continuity, fix k and consider $t, s \in [t_k, t_{k+1})$ with $\sigma(t) = i$:

$$|(\mathcal{T}\kappa)(t) - (\mathcal{T}\kappa)(s)| \leq A + B$$

where

$$\begin{aligned}
A &= |\zeta_i(t,\kappa(t)) - \zeta_i(s,\kappa(s))| \cdot |I^{\rho_i}\psi_i(t,\kappa(t))| \\
&\leq (a_i|t-s| + \phi_i(|\kappa(t) - \kappa(s)|))\,\frac{\|h_i\|_{L^1} t^{\rho_i - 1}}{\Gamma(\rho_i)} \\
&\leq (a_i|t-s| + \phi_i(H_\kappa|t-s|^{\varsigma}))\,\frac{\|h_i\|_{L^1} T^{\rho_i - 1}}{\Gamma(\rho_i)} \\
B &= |\zeta_i(s,\kappa(s))| \cdot |I^{\rho_i}\psi_i(t,\kappa(t)) - I^{\rho_i}\psi_i(s,\kappa(s))| \\
&\leq (\phi_i(|\kappa(s)|) + \bar{\zeta}_i)\frac{1}{\Gamma(\rho_i)}\int_0^T |(t-\tau)^{\rho_i - 1} - (s-\tau)^{\rho_i - 1}| h_i(\tau)d\tau \\
&\leq (\phi_i(\|\kappa\|_\infty) + \bar{\zeta}_i)\frac{\|h_i\|_{L^1}|t-s|^{\rho_i}}{\Gamma(\rho_i+1)}
\end{aligned}$$

Using Proposition 1, $||\kappa||_\infty \leq \max(1, T^\varsigma)||\kappa||_\varsigma$, so:

$$|(\mathcal{T}\kappa)(t) - (\mathcal{T}\kappa)(s)| \leq C_i|t-s|^{\min(\varsigma,\rho_i)}$$

Thus $\mathcal{T}\kappa \in H^\varsigma[0,T]$ for $\varsigma \leq \min_i \rho_i$.
For $\kappa \in B_{r_0} = \{\kappa : ||\kappa||_\varsigma \leq r_0\}$, at $t = 0$: $|(\mathcal{T}\kappa)(0)| = 0$. For $t > 0$:

$$\begin{aligned}|(\mathcal{T}\kappa)(t)| &\leq |\zeta_i(t,\kappa(t))| \cdot |I^{\rho_i}\psi_i(t,\kappa(t))| \\ &\leq (\phi_i(|\kappa(t)|) + \bar{\zeta}_i)\frac{1}{\Gamma(\rho_i)}\int_0^t (t-s)^{\rho_i-1}h_i(s)ds \\ &\leq (\phi_i(r_0) + \bar{\zeta}_i)\frac{||h_i||_{L^1}T^{\rho_i-1}}{\Gamma(\rho_i)}\end{aligned}$$

The Hölder constant satisfies:

$$\begin{aligned}H_{\mathcal{T}\kappa} \leq{}& \frac{1}{\Gamma(\rho_i)}\left(a_iT^{1-\min(\varsigma,\rho_i)} + \phi_i(r_0)\right)||h_i||_{L^1}T^{\rho_i-1} \\ &+ \frac{1}{\Gamma(\rho_i+1)}(\phi_i(r_0) + \bar{\zeta}_i)||h_i||_{L^1}T^{\rho_i}\end{aligned}$$

Thus $||\mathcal{T}\kappa||_\varsigma \leq r_0$ by (A4).
B_{r_0} is compact due to compact embedding $H^\varsigma[0,T] \hookrightarrow C[0,T]$ (Theorem 2.1 in [26]). Continuity of $\mathcal{T}$ follows from uniform convergence and the dominated convergence theorem applied to the integral operator. By Schauder's theorem, $\mathcal{T}$ has a fixed point.

4. UNIQUENESS

We now strengthen our assumptions to Lipschitz continuity, transforming the solution operator into a contraction. This refinement leverages the Banach Fixed-Point Theorem, providing not only uniqueness but also the foundation for stability analysis. The contractivity constant derived here will be pivotal for our Hyers-Ulam stability results.

(B1) $|\zeta_i(t,\kappa) - \zeta_i(s,y)| \leq a_i|t-s| + b_i|\kappa - y|$

(B2) $|\psi_i(t,\kappa) - \psi_i(t,y)| \leq G_i|\kappa - y|$

(B3) There exists $R > 0$ such that:

$$C_3 := \sum_{i=1}^{N} \frac{2}{\Gamma(\rho_i+1)}\left[b_i||h_i||_{L^1} + (R + \bar{\zeta}_i)G_i\right]\max(1, T^\varsigma) < 1 \tag{5}$$

Theorem 4. *Under (A1), (B1)-(B3), FHSDE* (2) *has a unique solution in* $B_R \subset H^\varsigma[0,T]$.

Proof. For $\kappa, y \in B_R$:

$$\begin{aligned}||\mathcal{T}\kappa - \mathcal{T}y||_\varsigma &= |(\mathcal{T}\kappa)(0) - (\mathcal{T}y)(0)| + H_{\mathcal{T}\kappa-\mathcal{T}y} \\ &= 0 + \sup_{t\neq s}\frac{|[(\mathcal{T}\kappa)(t) - (\mathcal{T}y)(t)] - [(\mathcal{T}\kappa)(s) - (\mathcal{T}y)(s)]|}{|t-s|^\varsigma}\end{aligned}$$

Decompose into switching intervals:

$$\leq \sum_{k=0}^{M} \sup_{t,s\in[t_k,t_{k+1})} \frac{|(\mathcal{T}\kappa)(t) - (\mathcal{T}y)(t) - (\mathcal{T}\kappa)(s) + (\mathcal{T}y)(s)|}{|t-s|^{\varsigma}}$$

For each interval $[t_k, t_{k+1})$ with mode i:

$$\begin{aligned}
&\left|(\mathcal{T}\kappa)(t) - (\mathcal{T}y)(t) - [(\mathcal{T}\kappa)(s) - (\mathcal{T}y)(s)]\right| \\
&\leq |\zeta_i(t,\kappa(t))I^{\rho_i}\psi_i(t,\kappa(t)) - \zeta_i(t,y(t))I^{\rho_i}\psi_i(t,y(t))| \\
&+ |\zeta_i(s,\kappa(s))I^{\rho_i}\psi_i(s,\kappa(s)) - \zeta_i(s,y(s))I^{\rho_i}\psi_i(s,y(s))| \\
&\leq 2 \sup_{u\in[t_k,t_{k+1}]} \left|\zeta_i(u,\kappa(u))I^{\rho_i}\psi_i(u,\kappa(u)) - \zeta_i(u,y(u))I^{\rho_i}\psi_i(u,y(u))\right|
\end{aligned}$$

Now analyze the supremum:

$$\begin{aligned}
&\left|\zeta_i(u,\kappa(u))I^{\rho_i}\psi_i(u,\kappa(u)) - \zeta_i(u,y(u))I^{\rho_i}\psi_i(u,y(u))\right| \\
&\leq |\zeta_i(u,\kappa(u)) - \zeta_i(u,y(u))| \cdot |I^{\rho_i}\psi_i(u,\kappa(u))| \\
&+ |\zeta_i(u,y(u))| \cdot |I^{\rho_i}\psi_i(u,\kappa(u)) - I^{\rho_i}\psi_i(u,y(u))| \\
&\leq b_i|\kappa(u) - y(u)| \cdot \frac{1}{\Gamma(\rho_i)} \int_0^u (u-\tau)^{\rho_i-1} h_i(\tau)d\tau \\
&+ |\zeta_i(u,y(u))| \cdot \frac{1}{\Gamma(\rho_i)} \int_0^u (u-\tau)^{\rho_i-1} |\psi_i(\tau,\kappa(\tau)) - \psi_i(\tau,y(\tau))|d\tau \\
&\leq b_i\|\kappa - y\|_\infty \frac{\|h_i\|_{L^1} u^{\rho_i-1}}{\Gamma(\rho_i)} + (\phi_i(|y(u)|) + \bar{\zeta}_i)G_i\|\kappa - y\|_\infty \frac{u^{\rho_i}}{\Gamma(\rho_i+1)} \\
&\leq \left[b_i\|h_i\|_{L^1}\frac{T^{\rho_i-1}}{\Gamma(\rho_i)} + (\phi_i(R) + \bar{\zeta}_i)G_i\frac{T^{\rho_i}}{\Gamma(\rho_i+1)}\right]\|\kappa - y\|_\infty
\end{aligned}$$

Since $\|\kappa - y\|_\infty \leq \max(1, T^{\varsigma})\|\kappa - y\|_{\varsigma}$:

$$\|\mathcal{T}\kappa - \mathcal{T}y\|_{\varsigma} \leq 2\sum_{i=1}^{N}\left[\frac{b_i\|h_i\|_{L^1}T^{\rho_i-1}}{\Gamma(\rho_i)} + \frac{(\phi_i(R) + \bar{\zeta}_i)G_iT^{\rho_i}}{\Gamma(\rho_i+1)}\right]\max(1, T^{\varsigma})\|\kappa - y\|_{\varsigma}$$

By (B3), the coefficient is $C_3 < 1$. Thus $\mathcal{T}$ is a contraction on B_R, and by Banach's theorem, has a unique fixed point.

5. HYERS-ULAM STABILITY

Definition 1. *The FHSDE* (2) *is Hyers-Ulam stable if* $\exists K > 0$ *such that for every* $\epsilon > 0$ *and* $\kappa \in H^{\varsigma}[0,T]$ *satisfying:*

$$\left|\kappa(t) - \zeta_{\sigma(t)}(t,\kappa(t))I^{\rho_{\sigma(t)}}\psi_{\sigma(t)}(t,\kappa(t))\right| \leq \epsilon, \quad t \in [0,T] \tag{6}$$

there exists solution y *of* (2) *with* $\|\kappa - y\|_{\varsigma} \leq K\epsilon$.

Theorem 5. *Under assumptions of Theorem 4, FHSDE* (2) *is Hyers-Ulam stable with* $K = (1 - C_3)^{-1}\max(1, T^{\varsigma})$.

Proof. Let κ satisfy (6). For all $t \in [0, T]$:

$$|\kappa(t) - (\mathcal{T}\kappa)(t)| \leq \epsilon$$

Thus:

$$||\kappa - \mathcal{T}\kappa||_\infty \leq \epsilon$$

Since $|| \cdot ||_\varsigma \geq || \cdot ||_\infty$, we have:

$$||\kappa - \mathcal{T}\kappa||_\varsigma \leq \max(1, T^\varsigma)\epsilon$$

For the exact solution $y = \mathcal{T}y$:

$$\begin{aligned} ||\kappa - y||_\varsigma &= ||\kappa - \mathcal{T}y||_\varsigma \\ &\leq ||\kappa - \mathcal{T}\kappa||_\varsigma + ||\mathcal{T}\kappa - \mathcal{T}y||_\varsigma \\ &\leq \max(1, T^\varsigma)\epsilon + C_3||\kappa - y||_\varsigma \end{aligned}$$

Solving for $||\kappa - y||_\varsigma$:

$$||\kappa - y||_\varsigma \leq \frac{\max(1, T^\varsigma)\epsilon}{1 - C_3} = K\epsilon$$

where $K = \max(1, T^\varsigma)(1 - C_3)^{-1} > 0$ since $C_3 < 1$.

Example 1. *Consider a 2-mode system with $T = 0.3$, $\rho_1 = 0.6$, $\rho_2 = 0.4$, switching at $t = 0.15$:*

$$\sigma(t) = \begin{cases} 1 & t \in [0, 0.15) \\ 2 & t \in [0.15, 0.3] \end{cases}$$

with

$$\zeta_1(t, \kappa) = 0.1(t + \ln(1 + \kappa^2)), \quad \psi_1(t, \kappa) = \frac{\sin(\pi t)\kappa}{2(1 + |\kappa|)}$$

$$\zeta_2(t, \kappa) = 0.1\sqrt{t} + 0.05|\kappa|, \quad \psi_2(t, \kappa) = \frac{\cos(\pi t)\kappa}{4(1 + |\kappa|)}$$

Parameters: $b_1 = 0.2$, $b_2 = 0.1$, $G_1 = 1$, $G_2 = 0.25$, $h_1(t) = 0.5$, $h_2(t) = 0.125$, $||h_1||_{L^1} = 0.075$, $||h_2||_{L^1} = 0.01875$, $\bar{\zeta}_1 = 0.03$, $\bar{\zeta}_2 = 0.02$. *Choose* $R = 0.1$ *and* $\varsigma = 0.4$ *(satisfying* $\varsigma \leq \min(\rho_1, \rho_2) = 0.4$*):*

$$\begin{aligned} C_3 = &\frac{2}{\Gamma(1.6)}[0.2 \times 0.075 + (0.1 + 0.03) \times 1] \max(1, 0.3^{0.4}) \\ &+ \frac{2}{\Gamma(1.4)}[0.1 \times 0.01875 + (0.1 + 0.02) \times 0.25] \max(1, 0.3^{0.4}) \end{aligned}$$

With $\Gamma(1.6) \approx 0.8935$, $\Gamma(1.4) \approx 0.8873$, *and* $0.3^{0.4} \approx 0.617$*:*

$$\begin{aligned} \textit{Term 1} &= \frac{2}{0.8935}[0.015 + 0.13] \times 1 \approx 2.238 \times 0.145 \approx 0.324 \\ \textit{Term 2} &= \frac{2}{0.8873}[0.001875 + 0.03] \times 1 \approx 2.254 \times 0.031875 \approx 0.072 \\ C_3 &\approx 0.324 + 0.072 = 0.396 < 1 \end{aligned}$$

Thus $K = \max(1, 0.3^{0.4})(1 - 0.396)^{-1} \approx 1 \times 1.657 \approx 1.657$.

The system defined in Example 1 provides explicit validation of our core theorems. With fractional orders $\rho_1 = 0.6$ and $\rho_2 = 0.4$ switching at $t_s = 0.15$, the computed contraction constant $C_3 \approx 0.396 < 1$ satisfies condition (B3) of Theorem 4, confirming solution uniqueness in $B_{0.1} \subset H^{0.4}[0, 0.3]$. The Hyers-Ulam stability constant $K \approx 1.657$ derived from $K = (1 - C_3)^{-1}$ verifies Theorem 5, guaranteeing that for any ϵ-approximate solution κ satisfying (6), the exact solution y satisfies $||\kappa - y||_{0.4} \leq 1.657\epsilon$. The Hölder exponent $\varsigma = 0.4$ satisfies the regularity requirement in Proposition 1, ensuring solution continuity across switching points as mandated by Lemma 1. Parameters $b_1 = 0.2$, $b_2 = 0.1$ and $G_1 = 1, G_2 = 0.25$ fulfill Lipschitz conditions (B1)-(B2), while the structure of ζ_i satisfies the invertibility requirement (A1). This configuration demonstrates the simultaneous satisfaction of existence (Theorem 3), uniqueness (Theorem 4), and stability (Theorem 5) conditions within our theoretical framework.

6. CONCLUSION

This paper has established a comprehensive theoretical framework for fractional hybrid switched differential equations in Hölder spaces, advancing the mathematical foundations for systems exhibiting memory-dependent dynamics with discontinuous mode transitions. Through the synthesis of fractional calculus, switched systems theory, and functional analysis in Hölder spaces, we have demonstrated the existence of solutions using Schauder's fixed-point theorem, proven solution uniqueness via Banach's contraction principle, and derived explicit Hyers-Ulam stability bounds through a direct fixed-point approach. Our results rigorously address switching compatibility at discontinuity points while establishing Hölder spaces as the natural functional setting for solution regularity. The theoretical framework is validated through an explicit computational example that satisfies all derived conditions, including a computable Hyers-Ulam stability constant $K \approx 1.657$, demonstrating the practical applicability of our results for robust control design in complex systems with memory-dependent switching. This work opens new research directions in fractional hybrid systems, including state-dependent switching mechanisms, non-local boundary conditions, and stochastic switching signals.

DECLARATIONS

Author Contribution

- Conceptualization; Data curation; Formal analysis: B. Zada
- Validation; Writing - original draft: I. Khan
- Writing - review editing: N. Hussain

Conflicts of Interest

The author declares no conflicts of interest.

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All data relevant to the study are included in the article.

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